COMPOSITION OPERATORS ON THE DIRICHLET SPACE AND RELATED PROBLEMS

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Abstract. In this paper we investigate the following problem: when a bounded analytic function φ on the unit disk $\mathbb{D}$, fixing 0, is such that $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in $\mathcal{D}$?, and consider the problem of characterizing the univalent, full self-maps of $\mathbb{D}$ in terms of the norm of the composition operator induced. The first problem is analogous to a celebrated question asked by W. Rudin on the Hardy space setting that was answered recently ([3] and [15]). The second problem is analogous to a problem investigated by J. Shapiro in [14] about characterization of inner functions in the setting of H^2.

Let $\mathbb{D}$ denote the unit disk in the complex plane. By a *self- map* of $\mathbb{D}$ we mean an analytic map such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. The *composition operator* induced by φ is the linear transformation C_φ defined by $C_\varphi(f) = f \circ \varphi$ in the space of the holomorphic functions on $\mathbb{D}$.

The composition operators have been studied in many settings, and in particular in functional Banach spaces (cf. the books [4], [13], the survey of recent developments [8], and the references therein). Recall that a functional Banach space is a Banach space of analytic functions (on the disk or other domains of $\mathbb{C}$ or $\mathbb{C}^n$) where the evaluation functionals are continuous. The goal of this theory is to obtain characterizations of operator-theoretic properties of C_φ by function-theoretic properties of the symbol φ. Conversely, operator-theoretic properties of C_φ could suggest, or help to understand certain phenomena about function-theoretic properties of φ.

Particular instances of functional Banach spaces are the Hardy space H^2, and the Bergman space A^2 of the unit disk. In these spaces, as a consequence of Littlewood's Subordination Principle, every self-map of $\mathbb{D}$ induces a bounded composition operator. A very interesting setting for studying such operators is the Dirichlet space. Recall that if $dA(z) = \frac{1}{\pi} dx\, dy = \frac{1}{\pi} r\, dr\, d\theta$, $(z = x + iy = re^{i\theta})$ denotes the normalized area Lebesgue measure on $\mathbb{D}$, the *Dirichlet space* $\mathcal{D}$ is the Hilbert space of analytic functions in $\mathbb{D}$ with a square integrable derivative, with the norm given by

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\, dA(z).$$

It is well known that $\mathcal{D}$ is a functional Hilbert space, and for each $w \in \mathbb{D}$ the function

$$K_w(z) = 1 + \log \frac{1}{1 - \overline{w}z},$$

is the reproducing kernel at w in the Dirichlet space, that is, for $f \in \mathcal{D}$ we have $\langle f, K_w \rangle_{\mathcal{D}} = f(w)$. It is easy to see that $\|K_w\|_{\mathcal{D}}^2 = \log \frac{1}{1-|w|^2}$.

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A self-map of $\mathbb{D}$ does not induce, necessarily a bounded composition operator on $\mathcal{D}$. An obvious necessary condition for it is that $\varphi = C_\varphi z \in \mathcal{D}$ which is not always the case. Actually this condition is not sufficient. A necessary and sufficient condition in order to φ to induce a bounded composition operator on $\mathcal{D}$ is given in terms of counting functions and Carleson measures (see [9] and the references in this paper).

Recall that the *counting function* $n_\varphi(w)$, $w \in \mathbb{D}$, associated to φ is defined as the cardinality of the set $\{z \in \mathbb{D} : \varphi(z) = w\}$ when the latter is finite and understood as the symbol ∞ otherwise, with the usual rules of arithmetics holding in relation to the Lebesgue integral.

We will make use of a change of variable formula for non-univalent functions: Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic function with counting function $n_\varphi(w)$, if $f : \mathbb{D} \to [0, \infty)$ is any Borel function, then

$$\int_{\mathbb{D}} f(\varphi(z))|\varphi'(z)|^2 \, dA(z) = \int_{\mathbb{D}} f(w) n_\varphi(w) \, dA(w).$$

This formula is a particular instance of the general change of variable formula in [4, Th. 2.32] (see also [5]). In particular we obtain, as noted in [5], that $\int_{\mathbb{D}} |\varphi'(z)|^2 \, dA(z) = \int_{\mathbb{D}} n_\varphi(w) \, dA(w)$. So, φ is in the Dirichlet space if and only if its counting function is an L^1 function.

In two recent papers, [10] and [11], M. Martín and D. Vukotić, studied composition operators on the Dirichlet space. In this note, based on results in those works, we consider related questions. In Section 1, we investigate the analogous on Dirichlet spaces to a problem proposed by W. Rudin in the context of Hardy spaces: When a bounded analytic functions φ on the unit disk $\mathbb{D}$ fixing 0 is such that $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in $\mathcal{D}$?, and in Section 2 we consider the problem of characterizing the univalent, full self-maps of $\mathbb{D}$ in terms of the norm of the composition operator induced. This problem, is analogous to the question asked and answered by J. Shapiro in [14] about inner functions in the H^2 setting.

We write $\mathcal{D}_0$ for the subspace of $\mathcal{D}$ of those function in $\mathcal{D}$ vanishing in 0, and use the notation $\|C_\varphi : \mathcal{H} \to \mathcal{H}\|$ in order to denote the norm of the composition operator induced on the space $\mathcal{H}$.

1. Orthogonal functions in the Dirichlet space.

The problem of describing the isometric composition operators acting in Hilbert spaces of analytic functions has been studied in several settings. Namely, it was proved by Nordgren in [12] that the composition operator C_φ induced on H^2 by φ, a holomorphic self-map of the unit disk, is an isometry on H^2 if and only if $\varphi(0) = 0$ and φ is an inner function (see also [4, p. 321]. In A^2 it is an straightforward consequence of the Schwarz Lemma that φ induces an isometric composition operator if and only if it is a rotation.

Recently, M. Martin and D. Vukotić showed in [11] that in $\mathcal{D}$, the Dirichlet space in the unit disk, the isometric composition operators are those induced by univalent full maps of the disk into itself that fixes the origin. Recall that a self-map of $\mathbb{D}$ is said a *full map* if $A[\mathbb{D} \setminus \varphi(\mathbb{D})] = 0$.

W. Rudin in 1988 (at an MSRI conference) proposed the following problem: If φ is a bounded analytic on the unit disk $\mathbb{D}$ such that $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in H^2, does φ must be a constant multiple of an inner function? C.

Sunbberg [15] and C. Bishop [3] solved independently the problem. In fact, they show that there exists a function φ such that φ is not an inner function and $\{\varphi^n\}$ is orthogonal in H^2.

As asserted by M. Martín y D. Vukotić in [11], their characterization of the isometric composition operators acting on $\mathcal{D}$ can be interpreted as follows: the univalent full maps of the disk that fix the origin are the Dirichlet space counterpart of the inner functions that fix the origin for the composition operators on H^2. We propose the following question: When a bounded analytic function φ on the unit disk $\mathbb{D}$ fixing 0 is such that $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in $\mathcal{D}$? Recall that a bounded analytic function on $\mathbb{D}$ is not necessarily in $\mathcal{D}$, then we assume in this context that φ is in $\mathcal{D}$ (and therefore, since $\mathcal{D} \cap H^\infty$ is an algebra, that $\{\varphi^n\}$ is in $\mathcal{D}$).

We are going to answer this question in the case when n_φ is essentially bounded, that is, there is a constant C so that $n_\varphi(w) \le C$ for all w except those in a set of area zero. Our result is analogous to a characterization given by P. Bourdon in [2] in the context of H^2: the functions that satisfy the hypotheses of the Rudin's problem are characterized as those maps φ such that their Nevanlinna counting function N_φ is essentially radial. Our assumption that n_φ is essentially bounded is clearly stronger that assuming that φ is only in the Dirichlet space and it possibly can be relaxed. The proof relies in the techniques of the proof given in [2].

Theorem 1.1. *Let φ be a self-map on $\mathbb{D}$ fixing 0. The set $\{\varphi^n : n = 0, 1, 2, \dots\}$ is orthogonal in $\mathcal{D}$ if and only if there is a function $g : [0, 1) \to [0, \infty)$ such that for almost every $r \in [0, 1)$, $n_\varphi(re^{i\theta}) = g(r)$ for almost every $\theta \in [0, 2\pi]$ (this is, n_φ is essentially radial).*

Proof. Suppose that n_φ is essentially radial. Let $n > m$ be nonnegative integers. We have

$$\begin{aligned}
\langle \varphi^n, \varphi^m \rangle_{\mathcal{D}} &= nm \int_{\mathbb{D}} \varphi(z)^{n-1} \overline{\varphi(z)^{m-1}} |\varphi'(z)|^2 \, dA(z) \\
&= nm \int_{\mathbb{D}} w^{n-1} \overline{w^{m-1}} n_\varphi(w) \, dA(w) \\
&= nm \int_0^1 r^{n+m-1} \left[\frac{1}{\pi} \int_0^{2\pi} e^{i(n-m)\theta} n_\varphi(re^{i\theta}) \, d\theta \right] dr \\
&= nm \int_0^1 r^{n+m-1} g(r) \left[\frac{1}{\pi} \int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \right] dr \\
&= 0.
\end{aligned}$$

Conversely, if $\{\varphi^n : n = 0, 1, 2 \dots\}$ is orthogonal in $\mathcal{D}$. Let k be an arbitrary positive integer. For each integer $n > k$, we have

$$\begin{aligned}
0 = \langle \varphi^n, \varphi^{n-k} \rangle_{\mathcal{D}} &= n(n-k) \int_{\mathbb{D}} \varphi(z)^{n-1} \overline{\varphi(z)^{n-k-1}} |\varphi'(z)|^2 \, dA(z) \\
&= n(n-k) \int_{\mathbb{D}} w^{n-1} \overline{w^{n-k-1}} n_\varphi(w) \, dA(w) \\
&= n(n-k) \int_0^1 r^{2n-k-1} \left[\frac{1}{\pi} \int_0^{2\pi} e^{ik\theta} n_\varphi(re^{i\theta}) \, d\theta \right] dr.
\end{aligned}$$

The functions $f_k(r) := \int_0^{2\pi} e^{ik\theta} n_\varphi(re^{i\theta}) \, d\theta$ are in $L^2[0, 1]$ since n_φ is essentially bounded (it is the only instance of this hypothesis) and the precedent equation says that they are orthogonal in $L^2[0,1]$ to $\{r \mapsto r^{2n-k-1} : n > k\}$. By an slight variation of Müntz-Szasz Theorem (cf. [2]), the linear span of this set is dense in $L^2[0,1]$, and so $f_k(r) = 0$ for almost every $r \in [0,1]$. Taking complex conjugates, we see that $\int_0^{2\pi} e^{ij\theta} n_\varphi(re^{i\theta})\,d\theta = 0$ for all $j \neq 0$, and almost every $r \in [0,1]$. Thus that $\theta \mapsto n_\varphi(re^{i\theta})$ is essentially constant for almost every r. □

The following Proposition describes the self-maps of $\mathbb{D}$ that share the properties in the condition of the the previous Theorem.

Proposition 1.2. *Suppose that φ is a self-map with counting function essentially bounded, and essentially radial. Then φ is a constant multiple of a full self-map of $\mathbb{D}$.*

Proof. Suppose that φ is not constant. If the range of φ contains a point in the circle $S_r = \{re^{i\theta} : \theta \in [0, 2\pi]\}$, $\varphi(\mathbb{D})$ contains an arc because this is an open subset of $\mathbb{D}$. In this arc $n_\varphi \geq 1$, and so the range of φ may omit only a θ-zero-measure subset of S_r because n_φ is essentially constant on S_r.

Thus the range of φ contain almost every point in the disk $\{z : |z| < \|\varphi\|_\infty\}$. □

2. What do composition operators know about full mappings?

In the Hardy space, J. Shapiro [14] has characterized, in terms of their norms, those composition operators C_φ whose symbol is an *inner function.* In fact, J. Shapiro showed:

(1) If $\varphi(0) = 0$ then φ is inner if and only if $\|C_\varphi : H_0^2 \to H_0^2\| = 1$, where H_0^2 is the subspace of functions in H^2 what vanish at 0, and
(2) If $\varphi(0) \neq 0$ then φ is inner if and only if $\|C_\varphi : H^2 \to H^2\| = \sqrt{\frac{1+|\varphi(0)}{1-|\varphi(0)|}}$.

We are going to investigate the analogous questions on the Dirichlet space.

In [10] M. Martín and D. Vukotić calculate the norm of the composition operator C_φ induced on $\mathcal{D}$ by a univalent full map φ of $\mathbb{D}$. They obtain

$$\|C_\varphi : \mathcal{D} \to \mathcal{D}\| = \sqrt{\frac{L + 2 + \sqrt{L(4+L)}}{2}}, \tag{2.1}$$

where $L = \log \frac{1}{1-|\varphi(0)|^2}$, and show that it is an upper bound on the norms of composition operators acting on the Dirichlet space induced by univalent symbols.

The results in [14] and the assertion in [11], mentioned previously, that the univalent full maps of the disk that fix 0 are the Dirichlet space counterpart of the inner functions that fix the origin for the composition operators on H^2, lead us to investigate if the equality in the equation (2.1) characterizes the univalent full maps of the disk inside the univalent self-maps of $\mathbb{D}$.

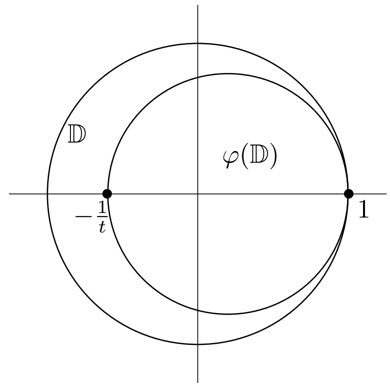
In addition, the main result in [11] says that $\varphi(0) = 0$ and φ is a univalent full self-map of the disk if and only if C_φ is an isometry on $\mathcal{D}$, and hence on $\mathcal{D}_0$, so in particular its restriction to $\mathcal{D}_0$ has norm 1. Is the converse true?

It is easy to see that this is not true. In fact, let φ_t, $t \geq 1$, be the linear fractional transformation given by

$$\varphi_t(z) = \frac{2z}{(1-t)z + (1-z)}, \quad z \in \mathbb{D}.$$

We easily see that $\varphi_t(\mathbb{D}) \subset \mathbb{D}$, $\varphi_t(0) = 0$, $\varphi_t(1) = 1$, and $\varphi_t(-1) = -1/t$ (see figure). If $t > 1$ clearly φ_t is not full, but a calculation in [1, Cor. 6.1] shows that

$\|C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0\| = 1$ when φ is a linear fractional self-map of $\mathbb{D}$ with a boundary fixed point.



Nevertheless, we have the following results, analogous to the results in [14].

Theorem 2.1. *Suppose φ is a univalent, holomorphic self-map of $\mathbb{D}$, with n_φ essentially radial and $\varphi(0) = 0$. Then φ is a full map if and only if*

$$\|C_\varphi : \mathcal{D}_0 \to \mathcal{D}_0\| = 1.$$

Proof. We saw before one direction. For the converse, suppose that φ is a univalent holomorphic self-map of $\mathbb{D}$, with n_φ essentially radial, $\varphi(0) = 0$, and that φ is not a full map.

We are going to show that the restriction of C_φ to $\mathcal{D}_0$ has norm < 1. We have that $\varphi(\mathbb{D})$ is contained in the disk $D(0, \rho) = \{z : |z| < \|\varphi\|_\infty = \rho\}$ with $A[\varphi(\mathbb{D}) \setminus D(0, \rho)] = 0$ and $0 < \rho < 1$ (cf. proof of Proposition 1.2.)

We write

$$g(r) := \frac{1}{\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta,$$

and since $|f'|^2$ is subharmonic in $\mathbb{D}$ then g is monotone increasing for $0 \le r < 1$. The change of variable formula gives

$$\begin{aligned} \|C_\varphi\|^2_{\mathcal{D}} &= \int_{\mathbb{D}} |f'(\varphi(z))|^2 \, |\varphi'(z)|^2 \, dA(z) \\ &= \int_{\varphi(\mathbb{D})} |f'(w)|^2 \, dA(w) \\ &= \int_0^\rho g(r) \, r \, dr. \end{aligned}$$

and so:

$$\begin{aligned}\|f\|_{\mathcal{D}}^2 = \int_{\mathbb{D}} |f'(w)|^2\, dA(w) &= \int_0^\rho g(r)\, r\, dr + \int_\rho^1 g(r)\, r\, dr\\ &\geq \int_0^\rho g\, dr + \frac{1-\rho^2}{2} g(\rho)\\ &= \int_0^\rho g\, dr + \frac{(1-\rho^2)/2}{\rho^2/2}(\rho^2/2) g(\rho)\\ &\geq \int_0^\rho g\, dr + \frac{(1-\rho^2)/2}{\rho^2/2}\int_0^\rho g(r)\, r\, dr\\ &= \left(1+\frac{(1-\rho^2)/2}{\rho^2/2}\right)\int_0^\rho g(r)\, r\, dr\\ &= \left(1+\frac{(1-\rho^2)/2}{\rho^2/2}\right)\|C_\varphi\|_{\mathcal{D}}^2,\end{aligned}$$

for each $f \in \mathcal{D}_0$. It yields the desired result: the restriction of C_φ to $\mathcal{D}_0$ has norm $\leq \nu = \left(1+\frac{(1-\rho^2)/2}{\rho^2/2}\right)^{-1/2} < 1$. □

In the next theorem, we consider the case $\varphi(0) \neq 0$. The proof follows nearly the one in [14, Th. 5.2]).

Theorem 2.2. *Suppose φ is a univalent, holomorphic self-map of $\mathbb{D}$ with n_φ essentially radial and $\varphi(0) \neq 0$. Then φ is a full map if and only if*

$$\|C_\varphi : \mathcal{D} \to \mathcal{D}\| = \sqrt{\frac{L+2+\sqrt{L(4+L)}}{2}},$$

where $L = \log 1/(1-|\varphi(0)|^2)$.

Proof. The necessity is part of [10, Th. 1]. For the converse, suppose that φ is a univalent, holomorphic self-map of $\mathbb{D}$ with n_φ essentially radial, such that $\varphi(0) = p \neq 0$, and φ is not a full map. We want to show that the norm of C_φ is strictly less that $\sqrt{\frac{L+2+\sqrt{L(4+L)}}{2}}$, where $L = \log 1/(1-p^2)$.

For this we consider α_p, the standard automorphism of $\mathbb{D}$ that interchanges p with the origin, this is

$$\alpha_p := \frac{p-z}{1-\overline{p}z}, \quad z \in \mathbb{D}.$$

We write $\varphi_p = \alpha_p \circ \varphi$, which is 0 in the origin. Since this function is a univalent, self map of $\mathbb{D}$ with counting function essentially radial, but it is not full, the Theorem 2.1 affirms that the restriction of the operator C_{φ_p} to $\mathcal{D}_0$ has norm $\nu < 1$.

Because α_p is self-inverse, $\varphi = \alpha_p \circ \varphi_p$, and so, for each $f \in \mathcal{D}$:

$$C_\varphi f = C_{\varphi_p}(f \circ \alpha_p) = C_{\varphi_p} f + f(p),$$

where $g = f \circ \alpha_p - f(p)$.

The function $C_{\varphi_p} g$ belong to $\mathcal{D}_0$ and thus:

$$\begin{aligned}\|C_\phi f\|_{\mathcal{D}} =& \|C_{\varphi_p} g\|_{\mathcal{D}}^2 + |f(p)|^2\\ &\leq \nu^2 \|g\|_{\mathcal{D}}^2 + |f(p)|^2\\ &= \nu^2 \|(C_{\alpha_p} f) - f(p)\|_{\mathcal{D}}^2 + |f(p)|^2.\end{aligned} \tag{2.2}$$

Since $\langle h, 1\rangle_{\mathcal{D}} = h(0)$ for each $h \in \mathcal{D}$,

$$\langle C_{\alpha_p} f, f(p)\rangle_{\mathcal{D}} = \overline{f(p)} C_{\alpha_p} f(0) = |f(p)|^2,$$

and we obtain

$$\begin{aligned}\|(C_{\alpha_p} f) - f(p)\|_{\mathcal{D}}^2 &= \|C_{\alpha_p} f\|_{\mathcal{D}}^2 - 2\Re\langle C_{\alpha_p} f, f(p)\rangle_{\mathcal{D}} + |f(p)|^2 \\ &= \|C_{\alpha_p} f\|_{\mathcal{D}}^2 - 2|f(p)|^2 + |f(p)|^2 \\ &= \|C_{\alpha_p} f\|_{\mathcal{D}}^2 - |f(p)|^2.\end{aligned}$$

This identity and the Equation (2.2) yield,

$$\|C_\alpha f\|_{\mathcal{D}}^2 \le \nu^2 \|C_{\alpha_p} f\|_{\mathcal{D}}^2 + (1-\nu^2)|f(p)|^2.$$

We know from [11, Th. 1] that $\|C_{\alpha_p} : \mathcal{D} \to \mathcal{D}\| = (L + 2 + \sqrt{L(4+L)})/2$, and we have the following estimate for $|f(p)|$:

$$|f(p)| \le \|f\|_{\mathcal{D}} \|K_p\|_{\mathcal{D}} = \sqrt{1+L}\|f\|_{\mathcal{D}},$$

then

$$\|C_\alpha f\|_{\mathcal{D}}^2 \le \left[\nu^2 \left(\frac{L+2+\sqrt{L(4+L)}}{2}\right) + (1-\nu^2)(1+L)\right] \|f\|_{\mathcal{D}}^2,$$

and $\delta = \left[\nu^2 \left(\frac{L+2+\sqrt{L(4+L)}}{2}\right) + (1-\nu^2)(1+L)\right] < 1$ because $p \neq 0$ and $L > 0$. □

2.1. **The essential norm.** Recall that the essential norm of an operator T in a Hilbert space $\mathcal{H}$ is defined as $\|T\|_e := \inf\{\|T - K\| : K \text{ is compact }\}$, this is, the essential norm of T is its norm in the Calkin algebra. It is well known [4] that in any Hilbert space of analytic functions, we have

$$\|C_\varphi\|_e = \lim_n \|C_\varphi R_n\|, \tag{2.3}$$

where R_n denotes the orthogonal projection of $\mathcal{H}$ onto $z^n\mathcal{H}$.

In [14], it is proved that a self-map $\varphi : \mathbb{D} \to \mathbb{D}$ is inner if and only if the essential norm of C_φ in the Hardy space is equal to $\sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}$. Because of the analogies presented here between inner functions and univalent full-maps, one might ask: Are full-maps characterized by the fact that the essential norm of C_φ in the Dirichlet space is equal to $\sqrt{\frac{L+2+\sqrt{L(4+L)}}{2}}$? where $L = \log \frac{1}{1-|\varphi(0)|^2}$. The answer is not, in fact *every univalent full-map has essential norm equal to 1 in the Dirichlet space*:

Theorem 2.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ a univalent full-map, then $\|C_\varphi\|_e = 1$ in the Dirichlet space.*

Proof. Suppose first that $\varphi(0) = 0$, then ([11]) C_φ is an isometry and equation 2.3 gives:

$$\|C_\varphi\|_e = \lim_n \{\sup_{\|f\|=1} \|R_n f\|\} = \lim_n \|R_n\| = 1.$$

If $\varphi(0) = p \neq 0$, then the function $\varphi_p := \alpha_p \circ \varphi$ is a univalent full-map fixing the origin and then for every function $f \in \mathcal{D}$ with $\|f\| = 1$ we have that $\|C_\varphi R_n f\| = \|C_{\alpha_p} R_n f\|$. Thus, $\|C_\varphi\|_e = \|C_{\alpha_p}\|_e$.

But in [6, Cor. 5.9], it is proved that the essential norm of any composition operator induced by an automorphism of $\mathbb{D}$ is equal to 1 and the result follows. □

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