COMPOSITION OPERATORS ON SPACES OF ENTIRE FUNCTIONS

GERARDO A. CHACÓN, GERARDO R. CHACÓN AND JOSÉ GIMÉNEZ

Abstract. In this paper we study composition operators on spaces of entire functions. We determine which entire functions induce bounded composition operators on the Paley-Wiener space, L^2_π, and on the $E^2(\gamma)$ spaces. In addition, we characterize compact composition operators on these spaces. We also study the cyclic properties of composition operators acting on L^2_π.

1. Introduction and preliminaries

Let $\text{Hol}\,(D)$ denote the space of all analytic functions in a domain $D \subset \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of D, and let $\mathcal{H}$ be a linear subspace of $\text{Hol}\,(D)$. If φ is an analytic self-map of D such that $f \circ \varphi$ belongs to $\mathcal{H}$ for all $f \in \mathcal{H}$, then φ induces a linear operator $C_\varphi : \mathcal{H} \to \mathcal{H}$ defined as $C_\varphi(f) := f \circ \varphi$.

C_φ is called the *composition operator* with symbol φ.

A Hilbert subspace $\mathfrak{H}$ of $\text{Hol}\,(D)$ is said to be a *functional Hilbert space* if for all $w \in D$, the evaluation functional:

$$\delta_w : \mathfrak{H} \to \mathbb{C}; \quad f \mapsto f(w),$$

is continuous. In this case, as a consequence of the Riesz representation theorem, for each $w \in D$, there exists a function $k_w \in \mathfrak{H}$ such that

$$f(w) = \langle f, k_w \rangle, \quad f \in \mathfrak{H}.$$

Eeach function k_w $(w \in D)$ is known as a *reproducing kernel* of $\mathfrak{H}$.

A straightforward application of the closed graph theorem shows that a holomorphic function $\varphi : D \to D$ induces a continuous operator $C_\varphi : \mathfrak{H} \to \mathfrak{H}$ if and only if $f \circ \varphi \in \mathfrak{H}$ for each $f \in \mathfrak{H}$.

In this paper we will study bounded composition operators acting on certain functional Hilbert spaces of entire functions. In a recent paper ([6]), composition operators acting on the Fock space of entire functions were studied.

We refer the reader to [3] or [13] for the background on entire functions that we will use here. As usual, given an entire function f, we can estimate

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its radial growth by means of the function

$$M_f(r) := \max_{|z|=r} |f(z)|.$$

We recall that an entire function f is said to be of *finite order* if the inequality $M_f(r) < \exp(r^k)$ holds for sufficiently large values of r, for some $k > 0$. The *order* of an entire function f of finite order is the greatest lower bound of those values of k for which this asymptotic inequality is satisfied.

An entire function f, of order ρ, is said to be of *finite type* if for some $A > 0$ the inequality $M_f(r) < e^{Ar^\rho}$ holds for sufficiently large values of r. The greatest lower bound of those values of A for which this asymptotic inequality is satisfied is called the *type* of the function f. Following [3], we will say that an entire function f, is of *exponential type* σ if it is of order $\rho \leq 1$ and type $\sigma \in (0, \infty)$.

Another interesting issue in this context, is the study of the cyclic properties of composition operators. See, for example [5] and [10]. A bounded operator T acting on the Hilbert space $\mathfrak{H}$ is called *cyclic* if there is a vector $x \in \mathfrak{H}$ whose orbit under T

$$\mathrm{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$$

have dense linear span. If the set of all scalar multiples of $\mathrm{Orb}(T, x)$ is dense in $\mathfrak{H}$, then T is called *supercyclic*, and if the $\mathrm{Orb}(T, x)$ itself is dense in $\mathfrak{H}$, then T is called *hypercyclic*. As usual, x is called a cyclic (resp. supercyclic, hypercyclic) vector for T.

In Section 2, we will study composition operators on the so called Paley-Wiener space, L^2_π. We will characterize bounded and compact composition operators on L^2_π. On the other hand, we will investigate some aspects of the cyclic behavior of these operators. In Section 3, we will study composition operators on the Hilbert spaces of entire functions $E^2(\gamma)$ studied in [7].

2. Composition Operators on the Paley-Wiener space

2.1. **The Paley-Wiener space.** The *Paley-Wiener space* L^2_π is the space of those entire functions of exponential type less or equal than π whose restriction to the real line belongs to the space $L^2(-\infty, \infty)$ (cf. [3, 13]). L^2_π with the norm given by

$$\|f\|^2_{L^2(-\infty,\infty)} = \int_{-\infty}^{\infty} |f(x)|^2 \, dx,$$

is a closed subspace of $L^2(-\infty, \infty)$ and thus, it is a Hilbert space. Furthermore, the inequality

$$|f(x + iy)| \leq \sqrt{\frac{2}{\pi}} e^{\pi(|y|+1)} \|f\|_{L^2(-\infty,\infty)},$$

shows that L^2_π is a functional Hilbert space. It can be shown (cf. [13]) that its reproducing kernels are given by

$$k_w(z) = \frac{\sin \pi(z - \overline{w})}{\pi(z - \overline{w})}, \quad (w \in \mathbb{C})$$

and that set $\{k_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis in L^2_π. Therefore by Parseval's identity,

$$\|f\|^2_{L^2(-\infty,\infty)} = \sum_{n\in\mathbb{Z}} |f(n)|^2.$$

As usual we will write

$$\operatorname{sinc} z := \frac{\sin \pi z}{\pi z}.$$

As a consequence of the well-known Paley-Wiener theorem, the Unitary Fourier transform (the usual Fourier transform normalized) $\hat{f} := \mathfrak{F}(f)$ of a function $f \in L^2_\pi$ is supported in $[-\pi, \pi]$. Each function $f \in L^2_\pi$ admits the representation

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(t) e^{itz}\, dt, \quad \hat{f} \in L^2(-\pi, \pi),$$

where $L^2(-\pi, \pi)$ denote the closed subspace of $L^2(-\infty, \infty)$ consisting of those functions which vanish a.e. outside of $(-\pi.\pi)$. Then,

$$L^2_\pi = \mathfrak{F}^{-1}(L^2(-\pi, \pi)).$$

In fact, the Parseval's identity

$$\|f\|_{L^2(-\infty,\infty)} = \|\hat{f}\|_{L^2(-\pi,\pi)},$$

shows that the Unitary Fourier transform $\mathfrak{F} : L^2_\pi \to L^2(-\pi, \pi)$ is an isometric isomorphism.

2.2. **Bounded operators and compact operators on L^2_π.** As it was pointed out in [1], composition operators on the Paley-Wiener space appear also in the field of *signal processing*. A function f in $L^2(-\infty, \infty)$ is considered as a *signal* (with "finite energy") and belongs to the Paley-Wiener space if its frequency domain (the domain of its Unitary Fourier transform) is limited to the band $[-\pi, \pi]$.

In this setting the symbol φ is called a *warping function* and the operator $f \mapsto f \circ \varphi$ is a *warping operator.* In [1] (cf. also [2], a shorter version published) the authors considered the case $\varphi : \mathbb{R} \to \mathbb{R}$. We will study the case in which φ is an entire function. In the first place, we will consider the problem of characterizing the symbols φ such that C_φ acts as bounded operator on L^2_π.

In order to characterize the bounded composition operators on the Paley-Wiener space we will use the following results proved in [15] (see also [1, 11]).

Theorem 2.2.1. *Let f and φ be entire functions with $\varphi(0) = 0$. Let $F = f \circ \varphi$. Then there is a constant $c \in (0, 1)$ such that*

$$M_F(r) \geq M_f(cM_\varphi(r/2)).$$

Theorem 2.2.2 (Pólya's Theorem)**.** *Let f and φ be entire functions such that $F = f \circ \varphi$ is of finite order. Then either*

(1) *φ is a polynomial and f is of finite order, or*
(2) *φ is not a polynomial (but a function of finite order) and f is of order* 0.

Pólya's theorem implies that if a warping function $\varphi : \mathbb{R} \to \mathbb{R}$ maps every bandlimited functions to bandlimited functions, then φ is affine (cf. [1, 2]). In the proof of the following lema we use the ideas given in [1] and [2].

Lemma 2.2.3. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be a holomorphic map (no null), if the operator C_φ maps L^2_π into itself then φ is an affine map.*

Proof. Suppose that C_φ maps L^2_π into itself, then the function $\operatorname{sinc} \circ \varphi$ is in L^2_π and since sinc has order exactly one, Theorem 2.2.2 implies that φ is a polynomial. Let n denotes the degree of φ. We are going to show that $n = 1$.

Without loss of generality we may assume that $\varphi(0) = 0$. Then there is a positive constant a such that $M_\varphi(r) \geq ar^n$ for r large, and by Theorem 2.2.1 there exists a constant c, $0 < c < 1$ such that

$$M_{f\circ\varphi}(r) \geq M_f(car^n/2^n),$$

for each function of order one in L^2_π. Let $0 < b < 1$. If the order of f is one, then there are arbitrarily large values of R for which the inequality $M_f(R) \geq \exp R^b$ holds. Thus, there are arbitrarily large values of r such that

$$M_f(car^n/2) \geq \exp(ca^b r^{nb}/2^{nb}).$$

If $f \circ \varphi$ is of order $\rho \leq 1$, then there exist constants A, B such that

$$M_{f\circ\varphi}(r) \leq A \exp(Br),$$

for all r. Thus, there are arbitrarily large values of r such that

$$\exp(ca^b r^{nb}/2^{nb}) \leq A \exp(Br).$$

It follows that $nb \leq 1$. Since b is any positive number less that one, we must have $n = 1$ ($n > 0$ because a constant function can not be a symbol). □

Theorem 2.2.4. *Let φ be a nonconstant entire function. The operator C_φ is bounded on L^2_π if and only if $\varphi(z) = az + b$, $(z \in \mathbb{C})$ with $0 < |a| \leq 1$, and $a \in \mathbb{R}$.*

Proof. It is easy to see that the order and type of entire functions are preserved by translations. The Plancherel-Pólya theorem (cf. [13, Section 7.4]) shows that

$$\int |f(x + s + it)|^2\, dx = \int |f(x + it)|^2\, dx \leq \|f\|_{L^2(-\infty,\infty)} e^{2\pi|t|}.$$

Therefore, the space of entire functions of exponential type $\leq \pi$ that belong to $L^2(-\infty,\infty)$ is invariant under translations.

Now,

$$\int |f(ax)|^2\,dx = (1/|a|)\int |f(x)|^2\,dx, \quad (a\in\mathbb{R});$$

on the other hand, if the order of $f(z)$ is ρ, then the order of $f(az)$ is also ρ while if the type of $f(z)$ is σ, then the type of $f(az)$ is $|a|^\rho\sigma$.

This shows, in addition, that if $C_\varphi(L^2_\pi)\subset L^2_\pi$ then $\varphi(z)=az+b$ with $0<|a|\leq 1$. In order to see that $a\in\mathbb{R}$ it suffices to show that if $\psi(z)=iz$, then the function $f(z)=\operatorname{sinc} z$ belongs to L^2_π but $f\circ\psi\notin L^2_\pi$. Indeed,

$$|\operatorname{sinc} ix| = \frac{|e^{\pi x}-e^{-\pi x}|}{2\pi x} \geq \frac{e^{\pi x}}{2\pi x}-\frac{e^{-\pi x}}{2\pi x}\geq \frac{e^{\pi x}}{2\pi x}-\frac{1}{2\pi}, \quad (x\geq 1),$$

and if $A>0$ is such that $\frac{e^{\pi x}}{2\pi x}-\frac{1}{2\pi}\geq 1$ when $x\geq A$ we have

$$\int_{-\infty}^{\infty}|\operatorname{sinc} ix|^2\,dx \geq \int_A^\infty |\operatorname{sinc} ix|^2\,dx \geq \int_A^\infty 1\,dx = \infty.$$

The proposition is proved. □

Corollary 2.2.5. *No bounded composition operator on L^2_π is compact.*

Proof. Clearly, the operator C_φ is compact in L^2_π if and only if the operator $C_{\varphi-\varphi(0)}$ is compact.

Since $f(n)\xrightarrow[n\to\pm\infty]{} 0$ for all $f\in L^2_\pi$, the sequence of orthonormal vectors $\{k_n\}_{n=0}^\infty$ converges weakly to zero. However, an easy computation shows that

$$\|C_{\varphi-\varphi(0)}(k_n)\| = 1/\sqrt{|a|}$$

for all n. Consequently, C_φ can not be compact. □

2.3. **The adjoint of a composition operator on L^2_π.** Recall that by the Paley-Wiener Theorem, L^2_π and $L^2(-\pi,\pi)$ are isometrically isomorphic via $\mathfrak{F}$. If $f\in L^2_\pi$ we have:

$$f(z)=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\hat f(t)e^{itz}\,dt,$$

with $\hat f\in L^2(-\pi,\pi)$. Thus, if $\varphi(z)=az+b,\ b=\lambda+i\eta,\ a\in\mathbb{R},\ 0<a\leq 1$ (for the sake of simplicity we may assume that $a>0$), then we have

$$\begin{aligned} C_\varphi(f)(z)=f(az+\lambda+i\eta) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\hat f(t)e^{itaz}e^{i\lambda t}e^{-\eta t}\,dt \\ &= \frac{1}{a\sqrt{2\pi}}\int_{-a\pi}^{a\pi}\hat f\left(\frac{x}{a}\right)e^{i\lambda\left(\frac{x}{a}\right)}e^{-\eta\left(\frac{x}{a}\right)}e^{ixz}\,dx. \end{aligned}$$

Therefore, the composition operator C_φ corresponds, via $\mathfrak{F}$, to the operator $\hat C_\varphi : L^2(-\pi,\pi)\to L^2(-\pi,\pi)$ defined as

$$(\hat{C}_\varphi\, g)(x) \;:=\; \frac{1}{a}\, g\left(\frac{x}{a}\right) \chi_{(-a\pi, a\pi)}(x) e^{i\lambda\left(\frac{x}{a}\right)} e^{-\eta\left(\frac{x}{a}\right)}, \tag{1}$$

Now let $f, g \in L^2(-\pi, \pi)$, then

$$\begin{aligned} \langle \hat{C}_\varphi\, g, f\rangle &= \frac{1}{2a\pi}\int_{-a\pi}^{a\pi} g\left(\frac{x}{a}\right) e^{i\lambda\frac{x}{a}} e^{-\eta\frac{x}{a}}\overline{f(x)}\, dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) e^{i\lambda t} e^{-\eta t}\overline{f(ta)}\, dt \\ &= \langle g, \hat{C}_\varphi^*\, f\rangle. \end{aligned}$$

Consequently, the operator C_φ^* corresponds, via $\mathfrak{F}$, to the operator

$$(\hat{C}_\varphi^*\, f)(x) \;=\; e^{-i\lambda x} e^{-\eta x} f(ax), \quad f \in L^2(-\pi, \pi). \tag{2}$$

Since $\mathfrak{F}$ is an isometry, we can compute C_φ^* acting on the Paley-Wiener space as follows: Let $f \in L^2_\pi$ and $\hat{f} \in L^2(-\pi, \pi)$ its respective Unitary Fourier transform, then since $0 < a \le 1$, we have $\hat{C}_\varphi^* \hat{f} \in L^2(-\pi, \pi)$ and $\mathfrak{F}^{-1}(\hat{C}_\varphi^* \hat{f}) \in L^2_\pi$. Now,

$$\begin{aligned} (\mathfrak{F}^{-1}(\hat{C}_\varphi^* \hat{f}))(z) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-i\lambda x} e^{-\eta x} \hat{f}(ax) e^{izx}\, dx \\ &= \frac{1}{a\sqrt{2\pi}}\int_{-\pi}^{\pi} \hat{f}(t) e^{-i\lambda\frac{t}{a}} e^{-\eta\frac{t}{a}} e^{iz\frac{t}{a}}\, dt \\ &= \frac{1}{a\sqrt{2\pi}}\int_{-\pi}^{\pi} \hat{f}(t) e^{it\left(\frac{z-\lambda+i\eta}{a}\right)}\, dt \\ &= \frac{1}{a} f\left(\frac{z-\lambda+i\eta}{a}\right). \end{aligned}$$

Therefore,

$$(C_\varphi^* f)(z) = \frac{1}{a} f\left(\frac{z-\lambda+i\eta}{a}\right), \quad f \in L^2_\pi. \tag{3}$$

As a consequence of the equation (3) we have the following

Proposition 2.3.1. *A bounded composition operator C_φ on L^2_π is normal if and only if $\varphi'(0) = 1$.*

Proof. Let $\varphi(z) = az + b$. Using (3) it is straightforward to check that $C_\varphi C_\varphi^* - C_\varphi^* C_\varphi = 0$ if and only if $b = 0$. □

2.4. **Cyclic behavior of composition operators on L^2_π.** We shall now use equation (1) in order to prove the following result.

Theorem 2.4.1. *No bounded composition operator on L^2_π is supercyclic.*

Proof. Let $\varphi(z) = az + (\lambda + i\eta)$ and suppose that C_φ is supercyclic. For the sake of simplicity we will assume $a > 0$ and $\eta > 0$ (in the general case, the modifications needed are straightforward). Then there exists a function $g \in L^2(-\pi, \pi)$, a sequence $\{\alpha_k\}$ in $\mathbb{C}$, and a sequence $\{n_k\}$ in $\mathbb{N}$ such that $\left\{\alpha_k \hat{C}_\varphi^{n_k}(g)\right\}$ converges in $L^2(-\pi, \pi)$ to the function $f \equiv 1$. By taking a subsequence, if necessary, we may assume that $\alpha_k \hat{C}_\varphi^{n_k}(g)(x) \to 1$, a.e. but it is easy to see, using (1) that $\lim_{k\to\infty} \alpha_k \hat{C}_\varphi^{n_k}(g)(x) = 0$ if $a < 1$.

If $a = 1$, then $C_\varphi(g)(x) = g(x)e^{i\lambda x}e^{-\eta x}$. That is, C_φ is the multiplication operator $M_h : g \mapsto hg$ on $L^2(-\pi, \pi)$ with $h(x) = e^{i\lambda x}e^{-\eta x}$. Now we are going to use the angle criterion for supercyclic vectors [10]. Let $g \in L^2(-\pi, \pi) \setminus \{0\}$. Then Bessel's inequality implies that for every $k \in \mathbb{Z}^+$

$$|\langle \frac{\chi_{(-\pi,0)}}{\sqrt{\pi}}, e^{-k\eta(\cdot)}|g|\rangle|^2 + |\langle \frac{\chi_{(0,\pi)}}{\sqrt{\pi}}, e^{-k\eta(\cdot)}|g|\rangle|^2 \le \|e^{-k\eta(\cdot)}g\|^2$$

and therefore there exists a constant $\epsilon > 0$, depending on g, such that for every $k \in \mathbb{Z}^+$,

$$\sqrt{\pi}\left(\int_{-\pi}^{\pi} e^{-2k\eta x}|g(x)|^2\, dx\right)^{\frac{1}{2}} > \int_0^{\pi} e^{-k\eta x}|g(x)|\, dx + \epsilon.$$

Thus

$$\sup_k \frac{\int_0^\pi e^{-k\eta x}|g(x)|\, dx}{\sqrt{\pi}\left(\int_{-\pi}^{\pi} e^{-2k\eta x}|g(x)|^2\, dx\right)^{\frac{1}{2}}} < \sup_k \frac{\int_0^\pi e^{-k\eta x}|g(x)|\, dx + \epsilon}{\sqrt{\pi}\left(\int_{-\pi}^{\pi} e^{-2k\eta x}|g(x)|^2\, dx\right)^{\frac{1}{2}}} \le 1.$$

Therefore,

$$\sup_k \frac{|\langle M_h^k g, \chi_{(0,\pi)}\rangle|}{\|M_h^k g\|\|\chi_{(0,\pi)}\|} < 1.$$

Consequently, g can not be a supercyclic vector for M_h. □

2.5. **Composition operators on L^p_π.** Let us consider now the general Paley-Wiener spaces L^p_π, $1 \le p < \infty$, i.e., L^p_π is the space of those entire functions of exponential type $\le \pi$ whose restrictions to the real line belongs to the space $L^p(-\infty, \infty)$.

In a standard way (cf. [6]) the boundedness and compactness of C_φ is closely related to the properties of a certain Carleson-type measure associated with φ. However, a careful revision of the proof of theorem 2.2.4 will show to the reader that the result can be easily generalized to the setting of L^p_π spaces. The next two results involving Carleson measures have interest by themselves.

In what follows, we will assume that $1 \le p \le q < \infty$.

We shall say that a measure μ on $\mathbb{R}$ is a (p, q)-*Carleson measure* (for the Paley-Wiener space) if the space L^p_π is boundedly contained in $L^q(\mu)$, that is: if the inclusion map $i : L^p_\pi \to L^q(\mu)$ is bounded. If the map i is compact, we will say that μ is a (p,q)-*vanishing* Carleson measure (for the Paley-Wiener space).

For a given entire function φ, the weighted pullback measure μ_φ on $\mathbb{R}$ is given by

$$\mu_\varphi(E) := l(\varphi^{-1}(E)),$$

for every Borel subset E of $\mathbb{R}$, where l denotes the arc-length measure. Since one can easily check that $\|C_\varphi(f)\|_{L^q(-\infty,\infty)} = \|f\|_{L^q(\mu_\varphi)}$ for every $f \in L^p_\pi$, we have that if the operator $C_\varphi : L^p_\pi \to L^q_\pi$ is bounded, then μ_φ is a (p,q)-Carleson measure. Furthermore, if $1 < p$ then C_φ is compact if and only if μ_φ is a (p,q)-vanishing Carleson measure.

In this section, we shall characterize both Carleson and vanishing Carleson measures on Paley-Wiener spaces. In [14], a similar result is mentioned.

Now, we list some well-known facts about Paley-Wiener spaces L^p_π $(1 \leq p < \infty)$ that we will require (cf. [13, 20.1]).

Remarks 2.5.1. (i) As a consequence of Plancherel-Pólya theorem and since $|f|^p$ is a subharmonic function, then for any $x, y \in \mathbb{R}$ we obtain

$$|f(x+iy)|^p \leq \frac{2}{\pi} e^{\pi(|y|+1)} \|f\|^p_{L^p(-\infty,\infty)}.$$

In particular, there is a constant K (depending on p, but neither on x nor on f) such that

$$|f(x)|^p \leq K \|f\|^p_{L^p(-\infty,\infty)}.$$

(ii) Let $\{\lambda_n\} \subset \mathbb{C}$ be a sequence and let H, δ be positive numbers such that

$$|\operatorname{Im} \lambda_n| \leq H < \infty, \quad |\lambda_n - \lambda_m| \geq \delta \quad \text{for } n \neq m.$$

Then, for each $f \in L^p_\pi$,

$$\sum_n |f(\lambda_n)|^p \leq C \|f\|^p_{L^p(-\infty,\infty)},$$

where the constant C depends on δ, and H but not on f.

(iii) If $1 < p < \infty$, then for any sequence $\{c_k\}$ in $l^p(\mathbb{Z})$ there exists a unique solution in L^p_π for the interpolation problem $f(k) = c_k$, $k \in \mathbb{Z}$. In addition, the norms of f and the corresponding $\{c_k\}$ are comparable, i.e. there exists constants m, M such that

$$m \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p} \leq \|f\|_{L^p(-\infty,\infty)} \leq M \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p},$$

for all functions $f \in L^p_\pi$. In particular, this last inequality implies that if $1 \leq p < q < \infty$, then $L^p_\pi \subset L^q_\pi$.

Actually, we only are going to need the result for a simpler problem: given $x_0 \in \mathbb{R}$ we want to find a function $f \in L^p_\pi$ such that $f(x_0) = 1$. This is obtained by considering the sequence $c_k = 0$ if $k \neq 0$ and $c_0 = 1$ and translating the function with $f(k) = c_k$ by x_0.

(iv) *Bernstein Inequality:* (cf. [3, 11.1.2]) If f is an entire function of exponential type τ and $|f(x)| \leq M$ for all $x \in \mathbb{R}$, then $|f'(x)| \leq M\tau$.

$D(x,r)$ will denote the usual open ball in $\mathbb{R}$ with center x and radius r.

Theorem 2.5.2. *Let $1 \leq p \leq q < \infty$. The following statements are equivalent:*

(a) *The measure μ is a (p,q)-Carleson measure.*
(b) *For each $r > 0$ there exists a constant C such that $\mu(D(x,r)) \leq C$ for all $x \in \mathbb{R}$.*
(c) *There exist $r > 0$ and a constant C such that $\mu(D(x,r)) \leq C$ for all $x \in \mathbb{R}$.*

Proof. Assertions (b) and (c) are clearly equivalent.

(c)⇒(a): Let $C > 0$ be such that $\mu(D(x,1)) < C$ for each $x \in \mathbb{R}$ and let $x_n \in [n, n+1]$ be such that $|f(x_n)| = \max_{[n,n+1]} |f(x)|$. Then, we decompose the sequence $\{x_n\}$ in two subsequences: $\{\lambda_n\}$ and $\{\lambda'_n\}$, with $|\lambda_n - \lambda_m| \geq 1/2$ and $|\lambda'_n - \lambda'_m| \geq 1/2$ if $n \neq m$. Now,

$$\begin{aligned}
\left(\int |f|^q \, d\mu\right)^{\frac{1}{q}} &= \left(\sum_{n=-\infty}^{\infty} \int_n^{n+1} |f|^q \, d\mu\right)^{\frac{1}{q}} \\
&\leq \left(\sum_{n=-\infty}^{\infty} |f(x_k)|^q \mu([n, n+1])\right)^{\frac{1}{q}} \\
&\leq C^{\frac{1}{q}} \left(\sum_{n=-\infty}^{\infty} |f(x_k)|^q\right)^{\frac{1}{q}} \\
&\leq C^{\frac{1}{q}} \left(\sum_{n=-\infty}^{\infty} |f(x_k)|^p\right)^{\frac{1}{p}} \\
&= C^{\frac{1}{q}} \left(\sum_{n=-\infty}^{\infty} |f(\lambda_k)|^p + \sum_{n=-\infty}^{\infty} |f(\lambda'_k)|^p\right)^{\frac{1}{p}} \\
&\leq C^{\frac{1}{q}} (2K)^{\frac{1}{p}} \|f\|_{L^p(-\infty,\infty)},
\end{aligned}$$

where the constant K is as in 2.5.1(ii).

(a)⇒(c): Let K and M be constants such that $|f(x)| \leq K\|f\|_{L^p(-\infty,\infty)} \leq KM \left(\sum_{k=-\infty}^{\infty} |f(k)|^p\right)^{1/p}$, for all $x \in \mathbb{R}$, $f \in L^p_\pi$ (cf. 2.5.1(i,iii)). Let $r > 0$ be such that $\pi KMr < 1$. If (c) is false, then for each $n \in \mathbb{N}$ there exist x_n in $\mathbb{R}$ such that $\mu(D(x_n, r) > n$. But, by (ii) above, for each $n \in \mathbb{N}$, there exists $f_n \in L^p_\pi$ such that $f_n(x_n) = 1$ and $\|f_n\|_{L^p(-\infty,\infty)} \leq M$, (cf. 2.5.1(iii)). Bernstein's Inequality (2.5.1 (iv)) implies then that $|f'_n(x)| \leq \pi KM$ for all $x \in \mathbb{R}, n \in \mathbb{N}$, and an application of the Mean Value Theorem gives:

$$1 - |f_n(x)| \leq |1 - f_n(x)| = |f_n(x_n) - f_n(x)| = |f'_n(\xi)|\,|x - x_n| \leq \pi KMr,$$

for all $x \in D(x_n, r)$ and some real number ξ in this ball.

Finally, we have

$$\int |f_n|^q \, d\mu \geq \int_{D(x_n,r)} |f_n|^q \, d\mu \geq (1 - \pi KMr)^q n,$$

which contradicts the fact that μ is a Carleson measure and hence there must exists a constant $C > 0$ such that $\|f_n\|_{L^q(\mu)} \leq C\|f_n\|_{L^p(-\infty,\infty)} \leq CM$ □

For vanishing Carleson measures, we have the following.

Theorem 2.5.3. *Let $1 < p \leq q < \infty$. The following statements are equivalent:*

(a) *The measure μ is a (p, q)-vanishing Carleson measure.*
(b) *For each $r > 0$ $\mu(D(x, r)) \to 0$ as $x \to \infty$.*
(c) *There exist $r > 0$ such that $\mu(D(x, r)) \to 0$ as $x \to \infty$.*

Proof. Again, assertions (b) and (c) are clearly equivalent.

(c)⇒(a): Let $\{f_n\}$ be a sequence in L^p_π weakly convergent to zero; this is, the sequence $\left\{\|f_n\|_{L^p(-\infty,\infty)}\right\}$ is bounded (without loss of generality we may assume that it is bounded by 1), and f_n converges uniformly on compact sets to zero. We shall see that $\int |f_n|^q \, d\mu \to 0$.

As in the proof of Theorem 2.5.2, for each n we choose the sequence $\{x_{n,k}\}_k$ as follows: $x_{n,k}$ will be a number in $[k, k+1]$ where $|f_n|$ reaches its greatest value. Then, we decompose this sequence in two subsequences $\{\lambda_{n,k}\}$ and $\left\{\lambda'_{n,k}\right\}$ so that $|\lambda_{n,k} - \lambda_{n,j}| \geq 1/2$ and $|\lambda'_{n,k} - \lambda'_{n,j}| \geq 1/2$ when $k \neq j$. By remark 2.5.1(ii), there exists a constant C such that $\sum_k |f_n(\lambda_{n,k})|^p \leq C\|f_n\|^p_{L^p(-\infty,\infty)}$ (and $\sum_k |f_n(\lambda'_{n,k})|^p \leq C\|f_n\|^p_{L^p(-\infty,\infty)}$) for all n. Then for all $n \in \mathbb{N}$:

$$\left(\sum_{k\in\mathbb{Z}} |f_n(x_{n,k})|^q\right)^{\frac{1}{q}} \leq \left(\sum_{k\in\mathbb{Z}} |f_n(x_{n,k})|^p\right)^{\frac{1}{p}} \leq (2C)^{\frac{1}{p}} \|f_n\|_{L^p(-\infty,\infty)} \leq (2C)^{\frac{1}{p}}.$$

Given $\varepsilon > 0$, we take $N \in \mathbb{N}$ such that $\mu(D(x, 1)) < \frac{\varepsilon^q}{(4C)^{\frac{q}{p}}}$ if $|x| \geq N$, and if n is large enough such that $|f_n(x)|^q \leq \frac{\varepsilon}{2\mu([-N,N])}$ for all x in the compact set $[-N, N]$, then

$$\int_N^N |f_n|^q \, d\mu \leq \frac{\varepsilon}{2}.$$

Hence

$$\left(\int_{|x|>N} |f_n|^q \, d\mu\right)^{\frac{1}{q}} \leq$$

$$\left(\sum_{k=N}^{\infty} |f_n(x_{n,k})|^q \mu([k,k+1]) + \sum_{k=N}^{\infty} |f_n(x_{n,-k-1})|^q \mu([-k-1,-k])\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{\varepsilon}{8C} \sum_{k=N}^{\infty} |f(x_{n,k})|^q + \frac{\varepsilon}{8C} \sum_{k=N}^{\infty} |f(x_{n,-k-1})|^q\right)^{\frac{1}{q}} \leq \varepsilon/2,$$

thus, $\int |f_n|^q \, d\mu < \varepsilon$ for such n.

(a)⇒(c): As in the proof of (a)⇒(c) in Theorem 2.5.2 (with the notation of that proof) we take $r > 0$ such that $\pi KMr < 1$. If (c) is not true, then there exists $\varepsilon > 0$ and a sequence $\{x_n\} \subset \mathbb{R}$ such that $x_n \to \infty$ and $\mu(D(x_n, r) \geq \varepsilon$. Of course, we can assume that these discs are disjoint. We take now $f_n \in L^p_\pi$ with $f_n(x_n) = 1$ and $\|f_n\|^p_{L^p(-\infty,\infty)} \leq M$.

Since the inclusion operator $i : L^p_\pi \to L^q(\mu)$ is compact, we can assume (by taking a subsequence if necessary) that $f_n \to f \in L^q(\mu)$.

Again, as in the proof of theorem 2.5.2, by using Bernstein Inequality and the mean value theorem, we see that $|f_n(x)| \geq 1 - \pi KMr$ in the disc $D(x_n, r)$ for each n, and then

$$\int |f_n|^q \, d\mu \geq \int_{D(x_n,r)} |f_n|^q \, d\mu \geq (1 - \pi KMr)^q \varepsilon.$$

Now, by observing that

$$\begin{aligned}
\|f_n - f\|_{L^q(\mu)} &\geq \left(\int_{D(x_n,r)} |f_n - f|^q \, d\mu\right)^{1/q} \\
&\geq \left(\int_{D(x_n,r)} |f_n|^q \, d\mu\right)^{1/q} - \left(\int_{D(x_n,r)} |f|^q \, d\mu\right)^{1/q} \\
&\geq (1 - \pi KMr)\varepsilon^{1/p} - \left(\int_{D(x_n,r)} |f|^q \, d\mu\right)^{1/q},
\end{aligned}$$

we obtain $\left(\int_{D(x_n,r)} |f|^q \, d\mu\right)^{1/q} \geq (1 - \pi KMr)\varepsilon^{1/p} - \|f_n - f\|_{L^q(\mu)}$, and since $\|f_n - f\|_{L^q(\mu)} \to 0$, then there exists $n \geq N$ such that

$$\int_{D(x_n,r)} |f|^q \, d\mu \geq \left(\frac{(1 - \pi KMr)\varepsilon^{1/p}}{2}\right)^q.$$

But

$$\int |f|^q \geq \sum_{n \geq N} \int_{D(x_n,r)} |f|^q \, d\mu,$$

which is a contradiction because $f \in L^q(\mu)$. □

An important consequence of these characterizations is that the Carleson (vanishing Carleson) condition is actually independent of p and q, $1 \leq p \leq q < \infty$. From this last result and from section 2.2 we may conclude the following

Theorem 2.5.4. *Let $1 < p \leq q < \infty$ and suppose that C_φ acts from L^p_π to L^q_π. Then*

(1) *C_φ is bounded if and only if $\varphi(z) = az + b$, with $0 < |a| \leq 1, a \in \mathbb{R}$.*
(2) *C_φ is not compact.*

3. Composition operators on the spaces $E^2(\gamma)$

3.1. **Spaces $E^2(\gamma)$.** According to Chan and Shapiro [7], an entire function $\gamma(z) = \sum \gamma_n z^n$ is called a *comparison function* if $\gamma_n > 0$ for all n and if the sequence of ratios γ_{n+1}/γ_n decreases to zero as $n \to \infty$. In case that the sequence $(n+1)\gamma_{n+1}/\gamma_n$ decreases monotonically to $\tau \geq 0$, then γ is said to be an *admissible comparison function.*

For each comparison function γ, we define $E^2(\gamma)$ as the Hilbert space of entire functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

such that

$$\|f\|^2_{2,\gamma} := \sum_{n=0}^{\infty} \gamma_n^{-2} |a_n|^2 < \infty.$$

In this case, the inner product of $E^2(\gamma)$ is given by

$$\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle := \sum_{n=0}^{\infty} \gamma_n^{-2} a_n \overline{b_n},$$

and the functions $e_n(z) := \gamma_n z^n$, $n = 0, 1, 2 \ldots$, form an orthonormal basis for $E^2(\gamma)$. Moreover, if $f(z) = \sum a_n z^n \in E^2(\gamma)$, then the following inequality [7, Prop. 1.4] shows that the spaces $E^2(\gamma)$ are functional Hilbert spaces.

$$\begin{aligned} |f(z)| &\leq \sum_{n=0}^{\infty} |a_n||z|^n = \sum_{n=0}^{\infty} \frac{|a_n|}{\gamma_n} \gamma_n |z|^n \\ &\leq \|f\|_{2,\gamma} \left(\sum_{n=0}^{\infty} \gamma_n^2 |z|^{2n} \right)^{1/2} \\ &\leq \|f\|_{2,\gamma} \gamma(|z|). \end{aligned}$$

In fact, the reproducing kernels of $E^2(\gamma)$ are given by

$$K_w(z) = \hat{\gamma}(\overline{w} z),$$

where $\hat{\gamma}(z) := \sum \gamma_n^2 z^n$.

From [7, Prop. 1.3], it follows that the order and type of comparison functions γ affects the behavior of the functions in the corresponding Hilbert space $E^2(\gamma)$. Actually, every element of $E^2(\gamma)$ has order and type no more than that of γ.

On the other hand, in [7, Prop. 1.3], it is proved that if $(n+1)\gamma_{n+1}/\gamma_n \downarrow \tau$, $\tau > 0$, then γ is of order one and type τ. The case in which $\tau = 0$ is more delicate. For example, if we take $\gamma_1(z) := 1 + \sum_{n=1}^{\infty} \left(\frac{e}{2n}\right)^{2n} z^n$, then γ_1 is clearly an admissible comparison function with $\tau = 0$, order 1/2 and type 1. However, the function $\gamma_2(z) := \sum_{n=0}^{\infty} e^{-n^2} z^n$ is also an admissible comparison function with $\tau = 0$ but of order zero, see [13, Chapter 1]. This shows that the order and type of an admissible comparison function γ is not determined by τ in this case.

3.2. **Bounded Composition Operators on $E^2(\gamma)$.** In [7], Chan and Shapiro showed that if the sequence $\{(n+1)\gamma_{n+1}/\gamma_n\}$ is bounded, then each translation operator is bounded on $E^2(\gamma)$. In this section, we will consider the problem of characterizing the symbols that induce bounded composition operators and compact composition operators on $E^2(\gamma)$ when $\tau > 0$.

First, we need to observe that for each $\sigma < \tau$, the function $f_\sigma(z) := e^{\sigma z}$ belongs to $E^2(\gamma)$. This follows from the fact that, for each n, the inequality $(n+1)\gamma_{n+1}/\gamma_n \geq \tau$ implies that $\gamma_n \geq \tau^n \gamma_0/n!$ and thus

$$\|f_\sigma\|_{2,\gamma}^2 = \sum_{n=0}^{\infty} \frac{1}{\gamma_n^2} \frac{\sigma^{2n}}{(n!)^2} \leq \frac{1}{\gamma_0^2} \sum_{n=0}^{\infty} \frac{(n!)^2}{\tau^{2n}} \frac{\sigma^{2n}}{(n!)^2} = \frac{1}{\gamma_0^2} \sum_{n=0}^{\infty} \left(\frac{\sigma}{\tau}\right)^{2n} < \infty.$$

Theorem 3.2.1. *Let φ be an entire function. The operator C_φ is bounded on $E^2(\gamma)$ if and only if $\varphi(z) = az + b$ with $|a| \leq 1$.*

Proof. Suppose that $f \in E^2(\gamma)$ is of non zero finite order. If $f \circ \varphi$ belongs to $E^2(\gamma)$, then it is of finite order, and by Pólya's Theorem (2.2.1), φ must be a polynomial. Let $\varphi(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$, $n \geq 1$. If $\sigma < \tau$, then $f_\sigma \circ \varphi(z) = \exp(\sigma(a_n z^n + \cdots + a_0))$ belongs to $E^2(\gamma)$, which is a function of order n and type $\sigma|a_n|$. Thus $n = 1$ and $\sigma|a_n| \leq \tau$. Since σ is arbitrary, it has to be $|a_n| \leq 1$.

In order to see that the condition is sufficient, it suffices to show that the symbol $\varphi(z) = az$, $|a| \leq 1$ induces a bounded composition operator on $E^2(\gamma)$, since every translation operator is bounded in $E^2(\gamma)$ (cf. [7]). Indeed, if $f(z) = \sum c_n z^n$ belongs to $E^2(\gamma)$ then,

$$\|C_\varphi f\|_{2,\gamma}^2 = \sum_{n=0}^{\infty} \gamma_n^{-2} |a|^{2n} |c_n|^2 \leq \sum_{n=0}^{\infty} \gamma_n^{-2} |c_n|^2 = \|f\|_{2,\gamma}^2.$$

□

3.3. **Compact Composition Operators on $E^2(\gamma)$.** In order to characterize the symbols that induce compact composition operators on $E^2(\gamma)$, we shall show first that this space is closely related to the space $\mathrm{Hol}\,(\mathbb{C}) \cap L^2(\mathbb{C}, e^{-2\tau|\cdot|}\, dA)$ where dA denotes the area measure on $\mathbb{C}$. The ideas and computations below follow those in [16, 6.8] where it was considered the space $E(e^z)$.

In general, $\mathfrak{H}_W$ is defined as the Hilbert space of entire functions such that

$$\|f\|_W^2 := \int_{\mathbb{C}} |f(z)|^2 W(|z|)\, dA(z) < \infty,$$

where W is a certain weight function.

Lemma 3.3.1. *Let $\gamma(z) = \sum \gamma_n z^n$ be an admissible comparison function such that $0 < \tau = \lim_{n\to\infty}(n+1)\gamma_{n+1}/\gamma_n$. The space $\mathfrak{H}_{W_\tau}$, where $W_\tau(|z|) := e^{-2\tau|z|}$ is contained in $E^2(\gamma)$, and there exists a constant K such that $\|f\|_{2,\gamma} \le K\|f\|_{W_\tau}$ for all $f \in \mathfrak{H}_{W_\tau}$.*

On the other hand, the space $E^2(\gamma)$ is contained in the space $\mathfrak{H}_{W_t}$, where $W_t(|z|) := e^{-2t|z|}$, $t > \tau$. In this case there exists a constant C such that $\|f\|_{W_t} \le C\|f\|_{2,\gamma}$ for all $f \in E^2(\gamma)$.

Proof. As pointed out before, $\lim_{n\to\infty}(n+1)\gamma_{n+1}/\gamma_n = \tau > 0$, implies that

$$\gamma_n \ge \tau^n \gamma_0/n! \qquad n = 0, 1, 2, \ldots. \tag{4}$$

Moreover, if $t > \tau$ then we can find a constant C (depending only on t) such that

$$\gamma_n < C\frac{t^n}{n!} \qquad n = 0, 1, 2, \ldots. \tag{5}$$

Now, if $W_t(|z|) = e^{-2t|z|}$, $t \ge \tau$, and $f(z) = \sum a_n z^n$, then

$$\|f\|_{W_t}^2 = 2\pi \sum_{n=0}^{\infty} p_n |a_n|^2$$

where $p_n := \displaystyle\int_0^\infty r^{2n+1} W_t(r)\, dr = (2n+1)!(2t)^{-2(n+1)}$ and thus

$$\frac{p_n}{(n!)^2} = (2n+1)\left[\frac{(2n)!}{(n!)^2}\right](2t)^{-2(n+1)}, \quad n = 0, 1, 2\ldots$$

The bracketed term is (by Stirling's formula) asymptotically a constant times $4^n/\sqrt{n}$, and therefore p_n is asymptotically a constant times $\sqrt{n}(n!)^2 t^{-2n}$.

If $t = \tau$ by using (4) we see that there is a constant C such that

$$\begin{aligned}
\|f\|_{W_\tau}^2 &= C\sum_{n=0}^{\infty} \sqrt{n}\frac{(n!)^2}{\tau^{2n}}|a_n|^2 \\
&\ge C\sum_{n=0}^{\infty} \sqrt{n}\frac{1}{\gamma_n^2}|a_n|^2 \\
&\ge C\sum_{n=0}^{\infty} \frac{1}{\gamma_n^2}|a_n|^2 = C\|f\|_{2,\gamma}^2.
\end{aligned}$$

On the other hand if $t > \tau$, we take $\tau < s < t$, and (5) gives:

$$\begin{aligned}\|f\|_{W_t} &= C\sum_{n=0}^{\infty}\sqrt{n}\frac{(n!)^2}{t^{2n}}|a_n|^2\\ &= C\sum_{n=0}^{\infty}\frac{(n!)^2}{s^{2n}}\sqrt{n}\left(\frac{s}{t}\right)^{2n}|a_n|^2\\ &\le C\sum_{n=0}^{\infty}\frac{1}{\gamma_n^2}\sqrt{n}\left(\frac{s}{t}\right)^{2n}|a_n|^2\\ &\le C\sum_{n=0}^{\infty}\frac{1}{\gamma_n^2}|a_n|^2 = C\|f\|_{2,\gamma}^2.\end{aligned}$$

□

The following lemma holds in any analytic functional Hilbert space. For the sake of completeness, we include here the proof for the case of $E_2(\gamma)$.

Lemma 3.3.2. *Let $\{f_n\}$ be a sequence in $E^2(\gamma)$. $\{f_n\}$ converges weakly to zero if and only if $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{C}$.*

Proof. The *if* part is inmediate. On the other hand, if $\{f_n\}$ converges weakly to zero, then it converges pointwise to zero (cf. [9]) and there exists $M > 0$ such that $\|f_n\|_{2,\gamma} \le M$ for all n. Let $K \subset \mathbb{C}$ compact, then for each $w \in K$ we have,

$$\begin{aligned}|f_n(w)| &= |\langle f_n, K_w\rangle| \le \|f_n\|_{2,\gamma}\|K_w\|\\ &= \|f_n\|_{2,\gamma}\hat{\gamma}(|w|^2) \le M\sup_{w\in K}\hat{\gamma}(|w|^2).\end{aligned}$$

Thus, $\{f_n\}$ is locally bounded and is therefore a normal family. If $\{f_n\}$ does not converge to zero uniformly on compacts subsets of $\mathbb{C}$, then there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ such that $\sup_{w\in K}|f_{n_k}(w)| > \epsilon$. But $\{f_{n_k}\}$ is a normal family and then it has a subsequence converging to zero uniformly on compact subsets of $\mathbb{C}$. This is a contradiction. □

Theorem 3.3.3. *Let φ be an entire function and $\gamma = \sum \gamma_n z^n$ an admissible comparison function such that $\lim(n+1)\gamma_{n+1}/\gamma_n = \tau > 0$. Then, C_φ is compact in $E^2(\gamma)$ if and only if $\varphi(z) = az + b$ where $|a| < 1$.*

Proof. Recall that the sequence $\{e_n\}$, $e_n(z) = \gamma(n)z^n$ is an orthonormal basis of $E^2(\gamma)$, in particular, $\{e_n\}$ is weakly convergent to zero. If $\varphi(z) = az$ with $|a| = 1$, then

$$\|C_\varphi e_n\|_{2,\gamma}^2 = \|a^n e_n\|_{2,\gamma}^2 = 1,$$

and C_φ can not be compact. It follows then that if $\varphi(z) = az + b$ ($|a| = 1$), then C_φ can not be compact either.

Now, we shall prove that if $\varphi(z) = az$, $|a| < 1$, then C_φ has to be compact. Let $\{f_n\}$ be a sequence in $E^2(\gamma)$ weakly convergent to zero. Then by lemma 3.3.2, $\{f_n\}$ is bounded by $M > 0$ and converges to zero uniformly on compact subsets of $\mathbb{C}$. We must prove that $\|f_n \circ \varphi\|_{2,\gamma}$ converges to zero.

By lemma 3.3.1, there exists a constant $c > 0$ such that

$$\|f\|^2_{2\gamma} \leq c \int |f(z)|^2 e^{-\tau|z|}\, dA(z).$$

Thus, is suffices to show that $\int |f_n(az)|^2 e^{-2\tau|z|}\, dA(z)$ converges to zero, or by changing variables that

$$\int |f_n(w)|^2 e^{-2\tau|w|/|a|}\, dA(w) \to 0.$$

Given $\varepsilon > 0$, we take $\tau < t < \tau/|a|$. Again be lemma 3.3.1, there exists a constant $C > 0$ such that

$$\int |f(w)|^2 e^{-t|w|}\, dA(w) \leq C\|f\|^2_{2,\gamma}.$$

Now we may take a compact set $K \subset \mathbb{C}$ such that $e^{-2\tau|w|/|a|}e^{t|w|} < \varepsilon/2CM$ for each w out of K; therefore,

$$\begin{aligned}\int_{\mathbb{C}\setminus K} |f_n(w)|^2 e^{-2\tau|w|/|a|}\, dA(z) &= \int_{\mathbb{C}\setminus K} |f_n(w)|^2 e^{-2\tau|w|/|a|} e^{2t|w|} e^{-2t|w|}\, dA(w)\\ &\leq \frac{\varepsilon}{2CM} \int_{\mathbb{C}\setminus K} |f_n(w)|^2 e^{-2t|w|/|a|}\, dA(z)\\ &\leq \frac{\varepsilon}{2CM} C\|f\|^2_{2,\gamma}\\ &\leq \frac{\varepsilon}{2CM} CM = \frac{\varepsilon}{2}.\end{aligned}$$

Finally, we may take n large enough such that, for each $w \in K$,

$$|f_n(w)| \leq \varepsilon/2 \int e^{-2\tau|z|}\, dA(z).$$

Then

$$\int_K |f_n(w)|^2 e^{-2\tau|w|/|a|}\, dA(w) \leq \varepsilon/2,$$

and this completes the proof. □

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Gerardo A. Chacón, Universidad de los Andes, Núcleo Táchira
E-mail address: gchacon@cantv.net, gchacon@ula.ve

Gerardo R. Chacón, Universidad de los Andes, Facultad de Humanidades y Educación
E-mail address: grchacon@ula.ve

José Giménez, Universidad de los Andes, Facultad de Ciencias
E-mail address: jgimenez@ula.ve