Reflection-symmetric, asymptotically flat solutions of the electrovacuum stationary axisymmetric Einstein Maxwell equations

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Abstract. It is shown that an asymptotically flat solution of the stationary axisymmetric electrovacuum Einstein Maxwell equations is reflection-symmetric, if and only if its Ernst's potentials on a portion of the positive z-axis extending to infinity, e_+ and f_+, obey $e_+(z)e_+^*(-z) = 1$ and $f_+(z) = -f_+^*(-z)e_+(z)$, where $*$ denotes complex conjugation. Moreover we present Ernsts potentials $\mathcal{E}$ and $\Phi(z, \rho)$ on the symmetry axis as a functions of the multipolar moments and present three possible solutions for modelling of the rapidly rotating neutron star exterior gravitational field.

1. Introduction

It is known that astrophysical objects possess reflection-symmetric, for that reason getting solutions such describes a reflection-symmetric source is very interesting and important. In this work we consider asymptotically flat electro vacuum solutions of Einstein Maxwell equations. In the last three decades a variety of solutions have been obtained and interpretated through the theory of multipole moments. This solutions have been obtained using several methods and generating techniques e.g. the Hauser-Ernst homogeneous Hilbert problem or Sibgatullin's method, this methods let us to construct the whole solution everywhere from the

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Ernst's potentials on the symmetry axis. It implies that we can predetermine the physical content of a solution by taking the right form on the axis (or a portion of it) and then use one of the existing generating techniques. This gives us the advantage that we can decide on the physical content of a solution from its behaviour on the axis, where the expressions for the Ernsts potentials is usually simpler than the whole expression.

Using results first obtained in the context of multipole moments theory we prove the following theorem, with the definitions $e(z) = \mathcal{E}(0, z)$, $e_+(z) = e(z : z > z_p)$ and $f(z) = \Phi(0, z)$, $f_+ = f(z : z > z_p)$ where z_p is a contant.

2. Ernst functions and moments

It's well know that a metric manifold ($\mathcal{M}$,h) is asymptotically flat if another manifold exists so $(\tilde{\mathcal{M}}, \tilde{h})$ that $\tilde{\mathcal{M}} = \mathcal{M} \cup \Lambda$ (inclusion of the infinite point, Λ) and the metric ones are related by a transformation as:

$$h_{ij} \rightarrow \tilde{h}_{ij} = \Omega^2 h_{ij}.$$

The conformal factor Ω should satisfy the following conditions: $\Omega|_\Lambda = \tilde{D}_i \Omega|_\Lambda = 0$ and $\tilde{D}_i \tilde{D}_j \Omega|_\Lambda = 2h_{ij}|_\Lambda$, where Λ is the point added to the initial manifold that represents infinity. We have the Ernst potencial

$$\mathcal{E} = f + i\Psi - \Phi\Phi^* \tag{1}$$

$$\Phi = A_4 + iA_3^{'} \tag{2}$$

where

$$A_{3,\rho}^{'} = \rho^{-1} f(A_{3,z} - \omega A_{4,z}) \tag{3}$$

$$A_{3,\rho}^{'} = \rho^{-1} f(A_{3,\rho} - \omega A_{4,\rho}) \tag{4}$$

Now, instead of $\mathcal{E}$ and Φ we use the complex Ernst potentials ξ, q, which are the analogues of Newtonian gravitational potential and Coulomb potential respectively,

$$\mathcal{E}=\frac{1-\xi}{1+\xi}, \qquad \Phi=\frac{q}{1+\xi}. \tag{5}$$

Ω transforms ξ and q potentials to

$$\tilde{\xi}=\Omega^{-1/2}\xi, \qquad \tilde{q}=\Omega^{-1/2}q. \tag{6}$$

By a coordinate transformation

$$\begin{aligned}\bar{\rho} &= \frac{\rho}{\rho^2+z^2},\\ \bar{z} &= \frac{z}{\rho^2+z^2},\\ \bar{\phi} &= \phi\end{aligned} \tag{7}$$

we bring infinity at the origin of the axes $(\bar{\rho},\bar{z})=(0,0)$. Then, by choosing the conformal factor to be

$$\Omega=\bar{r}^2=\bar{\rho}^2+\bar{z}^2, \tag{8}$$

the conformal metric in the new coordinates takes the following form

$$\tilde{h}_{ij}=\begin{pmatrix} e^{2\gamma} & 0 & 0\\ 0 & e^{2\gamma} & 0\\ 0 & 0 & \bar{\rho}^2\end{pmatrix}, \tag{9}$$

which is flat at $\bar{r}=0$, since $\gamma|_{\bar{r}=0}=0$. The tensorial multipole moments of the electrovacuum spacetime are computed by the recursive relation (see [8])

$$\begin{aligned}P^{(0)} &= \tilde{\xi},\\ P^{(1)}_i &= \tilde{\xi}_{,i},\\ P^{(n+1)}_{i_1 i_2\ldots i_{n+1}} &= \mathcal{C}[\tilde{\nabla}_{i_{n+1}}P^{(n)}_{i_1\ldots i_n}-\frac{1}{2}n(2n-1)R_{i_1 i_2}P^{(n-1)}_{i_3\ldots i_{n+1}}],\end{aligned} \tag{10}$$

for the geometry, and

$$\begin{aligned}Q^{(0)} &= \tilde{q},\\ Q^{(1)}_i &= \tilde{q}_{,i},\\ Q^{(n+1)}_{i_1 i_2\ldots i_{n+1}} &= \mathcal{C}[\tilde{\nabla}_{i_{n+1}}Q^{(n)}_{i_1\ldots i_n}-\frac{1}{2}n(2n-1)R_{i_1 i_2}Q^{(n-1)}_{i_3\ldots i_{n+1}}].\end{aligned} \tag{11}$$

for the electromagnetic field. The symbol $\tilde{\nabla}$ is used to denote the covariant derivative in the conformal space and it should not be confused with the same symbol used in [7]. The operator $\mathcal{C}$ denotes the operation "symmetrize over all free indices and take the trace-free part". Note that all tensors should be evaluated at Λ (at infinity).When a manifold $\mathcal{M}$ has an orthogonal symmetry plane to z-axis, the multipolar moments of even order are real and those of odd order are imaginary pure. The parts real and imaginary of the P moments correspond respectively at the rotational moments of mass and, these last ones without equivalent classic. On the other hand, the parts real and imaginary of moments Q are the electrical and magnetic moments, respectively. Due to axisymmetry, the components of these tensorial moments are multiples of the corresponding scalar moments which are the projection of the tensorial moments on the axis of symmetry. Following Fodor *et al* [8], the scalar moments are defined as

$$\begin{aligned} P_n &= \frac{1}{n!}\, \tilde{P}^{(n)}_{2\ldots 2}\Big|_{\Lambda} \\ Q_n &= \frac{1}{n!}\, \tilde{Q}^{(n)}_{2\ldots 2}\Big|_{\Lambda} . \end{aligned} \tag{12}$$

The metric h in Weyl-Papapetrou's coordinates [10] of manifold $\mathcal{M}$ has the form:

$$ds^2 = -F(dt - \omega d\phi)^2 + F^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right], \tag{13}$$

where F, ω, and γ are functions of ρ and z. Assuming that spacetime is asymptotically flat, $\gamma \to 0$, $F \to 1$, and $\omega \to 0$ at infinity. The metric functions are defined as

$$F = \mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi}, \tag{14}$$

$$\begin{aligned} \omega_{,\rho} &= -\,\rho F^{-2}\mathrm{Im}(\mathcal{E}_{,z} + 2\bar{\Phi}\Phi_{,z}), \\ \omega_{,z} &= \rho F^{-2}\mathrm{Im}(\mathcal{E}_{,\rho} + 2\bar{\Phi}\Phi_{,\rho}), \end{aligned} \tag{15}$$

$$\begin{aligned} \gamma_{,\rho} &= \frac{\rho}{4(\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi})^2}[(\mathcal{E}_{,\rho} + 2\bar{\Phi}\Phi_{,\rho})(\bar{\mathcal{E}}_{,\rho} + 2\Phi\bar{\Phi}_{,\rho}) \\ &\quad - (\mathcal{E}_{,z} + 2\bar{\Phi}\Phi_{,z})(\bar{\mathcal{E}}_{,z} + 2\Phi\bar{\Phi}_{,z})] - \frac{\rho(\Phi_{,\rho}\bar{\Phi}_{,\rho} - \Phi_{,z}\bar{\Phi}_{,z})}{\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi}}, \\ \gamma_{,z} &= \frac{\rho\mathrm{Re}[(\mathcal{E}_{,\rho} + 2\Phi\bar{\Phi}_{,\rho})(\bar{\mathcal{E}}_{,z} + 2\Phi\bar{\Phi}_{,z})]}{2(\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi})^2} - \frac{2\rho\mathrm{Re}(\bar{\Phi}_{,\rho}\Phi_{,z})}{\mathrm{Re}\,\mathcal{E} + \Phi\bar{\Phi}}, \end{aligned} \tag{16}$$

It is easy to show that the Einstein equation for $\mathcal{E}$ and Φ [7] take the following form for ξ and q

$$(\xi\xi^* - qq^* - 1)\nabla^2\xi = 2(\xi^*\boldsymbol{\nabla}\xi - q^*\boldsymbol{\nabla}q)\cdot\boldsymbol{\nabla}\xi \tag{17}$$

$$(\xi\xi^* - qq^* - 1)\nabla^2 q = 2(\xi^*\boldsymbol{\nabla}\xi - q^*\boldsymbol{\nabla}q)\cdot\boldsymbol{\nabla}q. \tag{18}$$

We can write ξ and q as power series of $\bar{\rho}$,$\bar{z}$:

$$\tilde{\xi} = \sum_{i,j=0}^{\infty} a_{ij}\bar{\rho}^i\bar{z}^j,$$

$$\tilde{q} = \sum_{i,j=0}^{\infty} b_{ij}\bar{\rho}^i\bar{z}^j, \tag{19}$$

(here an assumption about analyticity of $\tilde{\xi}$ and $\tilde{q}$ is made implicitly) where for the symmetry axis $a_{0,j} = m_j$ and $b_{0,j} = q_j$ and putting in 17 and 18 one gets [7]

$$\begin{aligned}(r+2)^2\, a_{r+2,s} = & -(s+2)(s+1)a_{r,s+2} + \\ & + \sum_{\substack{k+m+p=r \\ l+n+q=s}} (a_{kl}a^*_{mn} - b_{kl}b^*_{mn}) \times \\ & \times [a_{pg}(p^2+g^2-4p-5g-2pk-2gl-2) + \\ & + a_{p+2,g-2}(p+2)(p+2-2k) + \\ & + a_{p-2,g+2}(g+2)(g+1-2l)],\end{aligned} \tag{20}$$

and

$$\begin{aligned}(r+2)^2\, b_{r+2,s} = & -(s+2)(s+1)b_{r,s+2} + \\ & + \sum_{\substack{k+m+p=r \\ l+n+q=s}} (a_{kl}a^*_{mn} - b_{kl}b^*_{mn}) \times \\ & \times [b_{pg}(p^2+g^2-4p-5g-2pk-2gl-2) + \\ & + b_{p+2,g-2}(p+2)(p+2-2k) + \\ & + b_{p-2,g+2}(g+2)(g+1-2l)].\end{aligned} \tag{21}$$

Using equations 20 and 21 one may express the constants a_{ij} and b_{ij} in terms of where m_j and q_j respectively which shows that $\tilde{\xi}$ and $\tilde{q}$ are uniquely determinate by their values on the axis, we mean

$$\tilde{\xi}(\bar{\rho}=0) = \sum_{i=0}^{\infty} m_i\bar{z}^i, \qquad \tilde{q}(\bar{\rho}=0) = \sum_{i=0}^{\infty} q_i\bar{z}^i. \tag{22}$$

Moreover, equations 20 and 21 implies that $a_{ij} = 0$ and $b_{ij} = 0$ if i is odd which is necessary for $\tilde{\xi}$ and $\tilde{q}$ to be analitic at Λ [7].

3. Reflection-symmetric solutions and their form on the axis

Firstly from15 it is clear that if F, ω, A_3 and A_4 are reflection-symmetric then $\mathcal{E}(\rho,-z)=\mathcal{E}^*(\rho,z)$ and $\Phi(\rho,-z)=\Phi^*(\rho,z)$ hence also $\xi(\rho,-z)=\xi^*(\rho,z)$ and $q(\rho,-z)=q^*(\rho,z)$ and vice versa. Hence from 16 γ is reflection-symmetric and hence the entire metric is so. Moreover $\tilde{\xi}$ and $\tilde{q}$ satisfy $\tilde{\xi}(\rho,-z)=\tilde{\xi}^*(\rho,z)$ and $\tilde{q}(\rho,-z)=\tilde{q}^*(\rho,z)$and since $\tilde{\xi}$ and $\tilde{q}$ are analytic in a neighbourhood of Λ and Λ is in a sense the origin so its power series expansion coefficients a_{ij} and b_{ij} will satisfy the following condition:

Condition 1. a_{ij} and b_{ij} are real if j is even while a_{ij} and b_{ij} are purely imaginary if j is odd. A necessary and sufficient condition for a spacetime to be reflection symmetric might have been formulated in terms of multipole moments since by theorem 1 of [6] multipole moments define a spacetime uniquely. In fact, it is not difficult to see from 10 and 11 or 12 that such a condition is that the multipole moments P and Q satisfy P_n and Q_n real if n even, P_n and Q_n, pure imaginary if n odd. Lets consider the theorem 1 of [5] which garantize that if a spacetime is reflection-symmetric on the axis then it is so everywhere.

Theorem 1 If $m_j(=a_{0j})$ and $q_j(=b_{0j})$satisfy the condition for all j $\Longleftrightarrow$ a_{ij} satisfying the condition for all i,j. This show is very similar to the one presentred in [5] for vacuum case, for that reason here we dont show them.

Theorem 2 A stationary axisymmetric spacetime is reflection-symmetric $\Longleftrightarrow$ its Ernst potential on a portion of the positive z-axis extending to infinity, e_+ and f_+, obey the functional relations

$$e_+(z)e_+^*(-z)=1 \tag{23}$$

and

$$f_+(z)=-f_+^*(-z)e_+(z) \tag{24}$$

In order to proof 23 we can consider that $\tilde{\xi}(-\bar{z})=\tilde{\xi}^*(\bar{z})$, i.e.

$$-\frac{1}{\bar{z}}\frac{1-e_+(-\bar{z})}{1+e_+(-\bar{z})}=\frac{1}{\bar{z}}\frac{1-e_+^*(\bar{z})}{1+e_+^*(\bar{z})}$$

and hence $e_+(z)$ satisfies 23. Conversely if $e_+(z)$ satisfies 23 the operation $\bar{z} \rightarrow -\bar{z}$ has the effect of $\tilde{\bar{\xi}}(\bar{z}) \rightarrow \tilde{\bar{\xi}}(-\bar{z})$ Hence the coefficients of $\tilde{\bar{\xi}}$ satisfy the condition in the cross section of a neighbourhood of Λ and the positive part of the axis if and only if $e_+(z)$ satisfies 23. Here by the theorem 1 the proof is finished.

In order to proof 24 we can consider that $\tilde{q}(-\bar{z}) = \tilde{q}^*(\bar{z})$, i.e.

$$-\frac{2}{\bar{z}}\frac{f_+(-\bar{z})}{1+e_+(-\bar{z})} = \frac{2}{\bar{z}}\frac{f_+^*(\bar{z})}{1+e_+^*(\bar{z})}$$

and hence $f_+(z)$ satisfies 24. Conversely if $f_+(z)$ satisfies 24 the operation $\bar{z} \rightarrow -\bar{z}$ has the effect of $\tilde{q}(\bar{z}) \rightarrow \tilde{q}(-\bar{z})$ Hence the coefficients of $\tilde{q}$ satisfy the condition in the cross section of a neighbourhood of Λ and the positive part of the axis if and only if $f_+(z)$ satisfies 24. Here by the theorem 1 the proof is finished.

Theorem 3 A stationary axisymmetric spacetime with rational Ernst potential in Weyl-Papapetrou coordinates is reflection-symmetric $\Longleftrightarrow$ its Ernst potential on a portion of the positive z-axis extending to infinity, e_+ and f_+, have the form

$$e(z) = \frac{z^N + \sum_{j=1}^{N}(-1)^j P_j z^{N-j}}{z^N + \sum_{j=1}^{N} P_j^* z^{N-j}}, \tag{25}$$

$$f(z) = \frac{\sum_{j=1}^{N}(-1)^{j+1} L_j z^{N-j}}{z^N + \sum_{j=1}^{N} P_j^* z^{N-j}}, \tag{26}$$

with

$$P_k = D_k + iS_k, \qquad L_{2k+1} = V_k \qquad L_{2k} = iE_k \tag{27}$$

where $D_k,\ S_k,\ V_k,\ E_k, \in \mathbb{R}$.

Proof for the $e(z)$ potential

It's a double implication theorem, for that reason we have to make two proofs. For the vacuum case we have

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around of a neighbourhood of Λ on a portion of the positive z-axis extending to infinity, $\tilde{\xi} = \frac{1}{\bar{z}}\xi(\rho = 0, \bar{z})$, can be written as

$$\tilde{\xi}(\bar{z}) = \frac{\sum_{j=1}^{N}\{P_j^* - (-1)^j P_j\}\bar{z}^{j-1}}{2 + \sum_{j=1}^{N}\{P_j^* + (-1)^j P_j\}\bar{z}^j}, \tag{28}$$

If 25 describe a reflection-symmetry source then the coefficients a_{ij} must satisfy the *condition 1*. So, it's easy check that, in the numerator, if j is odd then $(-1)^j = -1$ and z^{j-1} is a even power of z and $P_j^* - (-1)^j P_j = P_j^* + P_j$ is obvious that it is a real number, then when z^{j-1} is a even power its coefficient is real. On the other hand when j is even $(-1)^j = 1$ and z^{j-1} it is a odd power of z and $P_j^* - (-1)^j P_j = P_j^* - P_j$ is obvious a purely imaginary number, so when z^{j-1} is a odd power the its coefficient is purely imaginary. In the denominator we find the same behavior, so we can affirm that $\tilde{\xi}$ satisfy the *condition 1* and using the *theorem 1* we have proof that 25 describes a reflection-symmetry space time.

$\Longrightarrow$

Lets consider a reflection-symmetry space time, it implies that the coefficients in the expansion of $\tilde{\xi}$ around Λ must satisfy the *condition 1*. We will study a particular case, we mind the case of rational potentials for these kind of space time, in general [15] we can write $e(z)$ as

$$\mathcal{E}(\rho = 0, z) = \frac{z^N + \sum_{l=1}^{N} a_l z^{N-l}}{z^N + \sum_{l=1}^{N} b_l z^{N-l}}. \tag{29}$$

Equivalently, as we proof, if a space time is reflection symmetry so its potential

$e(z)$ must be satisfy 23, then

$$\frac{z^N + \sum_{l=1}^{N}(-1)^l b_l^* z^{N-l}}{z^N + \sum_{l=1}^{N}(-1)^l a_l^* z^{N-l}} = \frac{z^N + \sum_{l=1}^{N} a_l z^{N-l}}{z^N + \sum_{l=1}^{N} b_l z^{N-l}}, \tag{30}$$

the last expression is satisfied when $b_l^* = (-1)^l a_l$ or equivalent when $a_l^* = (-1)^l b_l$. Using the last conditions over the coefficients and redefining a_j as P_j and from the last expression we can rewrite 29 as

$$e(z) = \frac{z^N + \sum_{j=1}^{N}(-1)^j P_j z^{N-j}}{z^N + \sum_{j=1}^{N} P_j^* z^{N-j}}, \tag{31}$$

so our expression 25 is correct and it describes a reflection-symmetry space time, and the proof is finished.

Proof for the $f(z)$ potential

Analogously to the last case,

$\Longleftarrow$

around of a neighbourhood of Λ on a portion of the positive z-axis extending to infinity, $\tilde{q} = \frac{1}{z} q(\rho = 0, \bar{z})$, can be written as

$$\tilde{q}(\bar{z}) = \frac{\sum_{j=1}^{N} (-1)^{j+1} L_j \bar{z}^{j-1}}{2 + \sum_{j=1}^{N}\{P_j^* + (-1)^j P_j\}\bar{z}^j}. \tag{32}$$

If 26 describe a reflection-symmetry source then the coefficients b_{ij} must satisfy the *condition 1*. So, it's easy check that, in the numerator, if j is odd then z^{j-1} is a even power of z and $(1)^{j+1} L_j$ from 27 is obvious that it is a real number, then when z^{j-1} is a even power its coefficient is real. On the other hand when j is even then z^{j-1} is a odd power of z and $(1)^{j+1}L_j$ from 27 is a purely imaginary number, so when z^{j-1} is a odd power the its coefficient is purely imaginary. In the denominator this behavior already was studied for the $e(z)$ case, so we can affirm that $\tilde{q}$ satisfy the *condition 1* and using the *theorem 1* we have proof that 26 describes a reflection-symmetry space time.

$\Longrightarrow$

Now in the electro vacuum case, so lets consider a reflection-symmetry space time, it implies that the coefficients in the expansion of $\tilde{\xi}$ and $\tilde{q}$ around Λ must satisfy the *condition 1*. For our case we'll suppose that the $e(z)$ potential is like 25 so we have to study the $f(z)$ potential which can be written as [15]

$$\Phi(\rho = 0, z) = \frac{\sum_{l=1}^{N} c_l z^{N-l}}{z^N + \sum_{l=1}^{N} b_l z^{N-l}}, \tag{33}$$

Equivalently, as we proof, if a space time is reflection symmetry so its potential $f(z)$ must be satisfy 24, then

$$\frac{\sum_{l=1}^{N} (-1)^{l+1}[(-1)^l c_l^*] z^{N-l}}{z^N + \sum_{l=1}^{N} (-1)^l b_l z^{N-l}} = \frac{\sum_{l=1}^{N} (-1)^{l+1} c_l z^{N-l}}{z^N + \sum_{l=1}^{N} b_l z^{N-l}}, \tag{34}$$

the last expression is satisfied when $c_l = (-1)^l c^*$. If we consider that $c_l = V_l + iE_l$ with V_l and $E_l \in \mathbb{R}$ so we have

$$-(-1)^l V = V \tag{35}$$

$$(-1)^l E = E, \tag{36}$$

the expression 35 is satisfies only when l is odd and the other one, we mind 36 is satisfies only when l is even, then c_l can not be a complex parameter, instead of they must be defined as

$$c_{2k+1} = V_{2k+1} \qquad c_{2k} = iE_{2k}, \tag{37}$$

we mind the odd c_j parameters must be reals and the even ones must be purely imaginary. If we redefine c_j as L_j we can write 33 as 26 for the reflection-symmetry case and our proof has finished.

The N value is determined by the number of solution arbitrary parameters, for example in the Kerr case $N = 1$ because we have two arbitrary parameters, mass and angular moment, $\tilde{N}$ values is null. A more extend discussion about N is presented in section 5. In order to describe reflection-symmetry source in the electro vacuum case we have proofed that the Ernst's potential 25 and 26 have the value for N. The coefficients P_r and L_r can be interpreted physically through relativistic Simon's multipolar moments of mass,rotation, electric charge and magnetic currents respectively. In the next sections we will describe how this parameters are related with multipolar moments.

4. Multipolar moments and parameters on the symmetric axis

It's very useful in order to construct a new solution if you have a relationship between the multipolar moments and the Ernst's potentials parameters because in this way you can construct a solution from a multipolar structure prescribed. From 27 we have that $P_k = D_k + iS_k, L_{2k+1} = V_k, L_{2k} = iE_k$, so to get a relationship between the multipolar moments and the potential parameters we have to find a relation between D_k, S_k, V_k, E_k and the multipolar moments. This relation is present through the next expressions.

$$M_n = -\sum_{l=1}^{n} J_{n-l} S_l - \sum_{r=2}^{n} M_{n-r} D_r + D_{n+1} \tag{38}$$

$$J_n = \sum_{l=1}^{n} M_{n-l} S_l - \sum_{r=2}^{n} J_{n-r} D_r - S_{n+1} \tag{39}$$

$$Q_n = -\sum_{l=1}^{n} B_{n-l} S_l - \sum_{r=2}^{n} Q_{n-r} D_r + V_{n+1} \tag{40}$$

$$B_n = \sum_{l=1}^{n} Q_{n-l}S_l - \sum_{r=2}^{n} B_{n-r}D_r - E_{n+1} \tag{41}$$

$$l = 1, 3, 5, 7, 9...(2n-1) \qquad r = 2, 4, 6, 8, ...n$$

where M_n, J_n, Q_n and B_n are the multipolar moments on the symmetric axis. The inverse relations can be easily derived.

5. How to get solutions to describe physical objects with prescribed multipolar moments

In order to explain how to choose 25 and 26 to describe physical objects we are going to use the *New generalization of the Kerr metric referring to a magnetized spinning mass* as an example. This solution [11] can be interpreted as an exact asymptotically flat generalizing of the well known Kerr metric to the case of a rotating mass possessing an arbitrary magnetic dipole moment. So, first we have to get the Kerr solution, its solution that describes a rotating object, and only the first mass and rotation multipolar moment are arbitrary, for that reason we have to choose P_1 as $m + ia$ and all the others P_j vanish. Now we have to introduce an arbitrary magnetic dipole moment, so we can choose $L_2 = ib$ or $E_2 = b$ and all the others E_j vanish, as the object that is being described hasn't arbitrary electrical moments so we fixed $V_j = 0$. Now we have to write the Ernst's potential, from 25 we get

$$e(z) = \frac{z - P_1}{z + P_1^*} = \frac{z - m - ia}{z + m - ia} \tag{42}$$

we can write the electromagnetic potential $f(z)$ as

$$f(z) = \frac{L_2 z}{z^2 + P_1^* z} = \frac{ibz}{z(z + m - ia).} \tag{43}$$

We can compare former expressions with the ones that presented in [11] and affirm that they are correct. Lets consider other example, it could be the Exterior gravitational field of a magnetized spinning source possessing and arbitrary massquadrupole moment [12] . This is an asymptotically flat solution which contains four independent parameters associated with the mass, angular momentum, mass quadrupole, and magnetic moments of a source. Its Ernsts potentials are given throught the following expressions

$$e(z) = 1 - \frac{2m}{z+m} - \frac{2ia}{(z+m)^2} - \frac{2c}{(z+m)^3}$$

$$f(z) = \frac{ib}{(z+m)^2} \tag{44}$$

These Ernst potentials can not be writing in the form which is presented in this work, we mind as 25 and 26, so we can immediately that this solution is not reflection symmetry. A physically interesting and important case is the description of the space time geometry around a rapidly rotating neutron star. In the literature we can find some interior numerical models for this kind of stars [16, 17], but we don't have a exterior solution that describes the same local properties of the interior numerical models. It's known that a theoretical model for a rapidly rotating neutron star must posses four arbitrary parameters, we mind this model must describe a source with mass, angular moment, deformation and differential rotations arbitraries. It implies that we have to construct solutions which have the four first multipolar moments as arbitraries parameters. From our representation 25 we could obtain three vacuum solutions that have this properties, its Ernst's potential and multipolar moments are given by the next expressions

5.1. First solution

$$e(z) = \frac{z^2 - (m+ia)z + k + ib}{z^2 - (m+ia)z + k + ib}, \tag{45}$$

where

$$P_1 = m + ia \qquad P_2 = k + ib$$

it's first mass and rotation multipolar moments are given by

$$M_0 = m \qquad M_1 = 0 \qquad M_2 = ab - m(a^2 + k) \qquad M_3 = 0$$
$$J_0 = 0 \qquad J_1 = -b + am \qquad J_2 = 0 \qquad J_3 = b(a^2 + k) - m(a^3 + 2ak).$$

This first solutions have already obtained by Manko, Martn & Ruz in 1995. They proposed the next choosing for the Ernst's potential

$$e(z) = \frac{(z - m - ia)(z + ib) - k}{(z + m - ia)(z + ib) - k}, \tag{46}$$

it's easy to check that this solution corresponding to the one found in this work if we redefine the potential parameters as

$$P_1 = m + i(a - b) \qquad P_2 = ab - k - ibm.$$

Moreover, this solution was recalculated by Manko, Sanabria-Gómez & Manko O V in 2000 using the next form

$$e(z) = \frac{(z - m - ia)(z + ib) + d - \delta - ab}{(z + m - ia)(z + ib) + d - \delta - ab}, \tag{47}$$

where

$$\delta = -\frac{m^2 b^2}{m^2 - (a - b)^2}, \qquad d = \frac{1}{4}[m^2 - (a - b)^2],$$

it's first mass and rotation multipolar moments are given by

$$M_0 = m \qquad M_1 = 0 \qquad M_2 = -m(a^2 - ab + d - \delta) \qquad M_3 = 0$$
$$J_0 = 0 \qquad J_1 = am \qquad J_2 = 0$$
$$J_3 = -m(a^3 + a(b^2 + 2d - 2\delta) - b(2a^2 + d - \delta)).$$

again it's very simple check that this solution corresponding to the one that obtained for us when

$$P_1 = m + i(a - b) \qquad P_2 = d - ibm - \delta.$$

This solution, or better the solutions that can be obtained from 47 using Sibgatullin's method[14], was reported as a exact solution for the exterior field of a rotating neutron star and as a solution that is able to describe the exterior field of a slowly or rapidly rotating neutron star [18]. This one was studied by Berti & Stergioulas [17] and some good results were obtained.

5.2. Second solution

$$e(z) = \frac{z^3 - (m + ia)z^2 + kz - ib}{z^3 + (m - ia)z^2 + kz - ib}, \tag{48}$$

where

$$P_1 = m + ia, \qquad P_2 = k, \qquad P_3 = ib$$

it's first mass and rotation multipolar moments are given by

$$\begin{array}{llll} M_0 = m & M_1 = 0 & M_2 = -(a^2 - k)m & M_3 = 0 \\ J_0 = 0 & J_1 = am & J_2 = 0 & J_3 = -m(a^3 - 2ak + s) \end{array}$$

5.3. Third solution

$$e(z) = \frac{z^4 - (m + ia)z^3 + k^2 z - ib}{z^4 + (m - ia)z^3 + k^2 z - ib}, \tag{49}$$

where

$$P_1 = m + ia, \qquad P_2 = k, \qquad P_3 = 0, \qquad P_4 = ib$$

it's first mass and rotation multipolar moments are given by

$$\begin{array}{llll} M_0 = m & M_1 = 0 & M_2 = -(a^2 - k)m & M_3 = 0 \\ J_0 = 0 & J_1 = am & J_2 = 0 & J_3 = -b - m(a^3 + 2ak) \end{array}$$

The last two solutions haven't reported in the literature and are being studied by the authors and others. For generating new physical solutions can we can proceed in the same way.

6. Conclusions

We have derived the conditions which Ernst's potential have to satisfy in order to describe reflection-symmetry sources and found the general form for this potentials in the rational case and the relationships of its parameters and the multipolar moments, all in the electro vacuum case for the stacionary axisymmetric Einstein

Maxwell equations. Moreover, using 38,39,40 and 41 we present a effective method to choose the Ernst's potential in order to describe physical objects with prescribe arbitrary multipolar moments. In addition we present a some solutions the ones that are perfect candidates to describe the rapidly rotating neutron star exterior gravitational field ,we mind the solutions which can be obtained from 45, 48 and 49.

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