GRAVITATIONAL RADIATION, VORTICITY AND THE ELECTRIC AND MAGNETIC PART OF WEYL TENSOR

June 21, 2005

1 The Bondi's formalism

$$\begin{aligned} ds^2 &= \left(\frac{V}{r}e^{2\beta} - U^2r^2e^{2\gamma}\right)du^2 + 2e^{2\beta}dudr \\ &+ 2Ur^2e^{2\gamma}dud\theta - r^2\left(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2\right) \end{aligned} \quad (1)$$

Regularity conditions: $\sin\theta -> 0$

$$V, \beta, U/\sin\theta, \gamma/\sin^2\theta \quad (2)$$

each equals a function of $\cos\theta$ regular on the polar axis.

$$\gamma = c(u,\theta)r^{-1} + \left(C(u,\theta) - \frac{1}{6}c^3\right)r^{-3} + ... \quad (3)$$

$$U = -\left(c_\theta + 2c\cot\theta\right) r^{-2} + \left[2N(u,\theta) + 3cc_\theta + 4c^2\cot\theta\right] r^{-3}... \tag{4}$$

$$\begin{aligned} V &= r - 2M(u,\theta) \\ &- \left(N_\theta + N\cot\theta - c_\theta^2 - 4cc_\theta\cot\theta - \frac{1}{2}c^2(1 + 8\cot^2\theta)\right) r^{-1} \\ &+ ... \end{aligned} \tag{5}$$

$$\beta = -\frac{1}{4}c^2 r^{-2} + ... \tag{6}$$

where letters as subscripts denote derivatives, and

$$4C_u = 2c^2 c_u + 2cM + N\cot\theta - N_\theta \tag{7}$$

$$M_u = -c_u^2 + \frac{1}{2}\left(c_{\theta\theta} + 3c_\theta\cot\theta - 2c\right)_u \tag{8}$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u\cot\theta + c_u c_\theta \tag{9}$$

$$m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta d\theta \tag{10}$$

$$m_u = -\frac{1}{2}\int_0^\pi c_u^2\sin\theta d\theta \tag{11}$$

1. If γ, M and N are known for some $u = a$(constant) and c_u (the news function) is known for all u in the interval $a \leq u \leq b$, then the system is fully determined

in that interval. In other words, whatever happens at the source, leading to changes in the field, it can only do so by affecting c_u and viceversa. At the light of this comment the relationship between news function and the occurrence of radiation becomes clear.

2. As it follows from (11), the mass of a system is constant if and only if there are no news.

$$u^{\alpha} = \left(\frac{1}{A}, 0, 0, 0\right) \tag{12}$$

$$A \equiv \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right)^{1/2}. \tag{13}$$

$$\omega^{\alpha} = \left(0, 0, 0, \omega^{\phi}\right) \tag{14}$$

$$\begin{aligned} \omega^{\phi} &= -\frac{e^{-2\beta}}{2r^2 \sin\theta}\{2\beta_{\theta}e^{2\beta} - \frac{2e^{2\beta}A_{\theta}}{A} - \left(Ur^2e^{2\gamma}\right)_r \\ &+ \frac{2Ur^2e^{2\gamma}}{A}A_r + \frac{e^{2\beta}\left(Ur^2e^{2\gamma}\right)_u}{A^2} \\ &- \frac{Ur^2e^{2\gamma}}{A^2}2\beta_u e^{2\beta}\} \end{aligned} \tag{15}$$

$$\Omega \equiv (-\omega_{\alpha}\omega^{\alpha})^{1/2} = \frac{e^{-2\beta-\gamma}}{2r}\{2\beta_{\theta}e^{2\beta} - 2e^{2\beta}\frac{A_{\theta}}{A} - \left(Ur^2e^{2\gamma}\right)_r$$

$$+ \quad 2Ur^2e^{2\gamma}\frac{A_r}{A} + \frac{e^{2\beta}}{A^2}\left(Ur^2e^{2\gamma}\right)_u - 2\beta_u\frac{e^{2\beta}}{A^2}Ur^2e^{2\gamma}\} \qquad (16)$$

$$\Omega = -\frac{1}{2r}(c_{u\theta} + 2c_u\cot\theta)$$
$$\frac{1}{r^2}\left[M_\theta - M(c_{u\theta} + 2c_u\cot\theta) - cc_{u\theta} + 6cc_u\cot\theta + 2c_uc_\theta\right] \qquad (17)$$

$$c_{u\theta} + 2c_u\cot\theta = 0 \qquad (18)$$

$$c_u = \frac{F(u)}{\sin^2\theta} \qquad (19)$$

$$F(u) = 0 \Longrightarrow c_u = 0 \qquad (20)$$

Let us now assume that initially (before some $u = u_0$ =constant) the system is static, in which case

$$N_u = c_u = 0 \qquad (21)$$

$$M_\theta = 0 \qquad (22)$$

and $\Omega = 0$ (actually, in this case $\Omega = 0$ at any order) as expected for a static field . Then let us suppose that at

$u = u_0$ the system starts to radiate ($c_u \neq 0$) until $u = u_f$, when the news vanish again. For $u > u_f$ the system is not radiating although (in general) $M_\theta \neq 0$ implying time dependence of metric functions (non-radiative motions).

For $u > u_f$ there is a vorticity term of order $\frac{1}{r^2}$ describing the effect of the tail of the wave. This in turn provides "observational" evidence for the violation of the Huygens's principle, a problem largely discussed in the literature.

2 The magnetic and electric part of Weyl tensor

$$E_{\alpha\beta} = C_{\alpha\gamma\beta\delta} u^\gamma u^\delta \tag{23}$$

$$H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \epsilon_{\alpha\gamma\epsilon\delta} C^{\epsilon\delta}{}_{\beta\rho} u^\gamma u^\rho, \qquad \epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{-g} \;\; \eta_{\alpha\beta\gamma\delta} \tag{24}$$

$\eta_{\alpha\beta\gamma\delta} = +1$ for $\alpha, \beta, \gamma, \delta$ in even order, -1 for $\alpha, \beta, \gamma, \delta$ in odd order and 0 otherwise. Also note that

$$\sqrt{-g} = r^2 \sin\theta e^{2\beta} \approx r^2 \sin\theta e^{-\frac{c^2}{2r^2}} \approx r^2 \sin\theta + O(1)$$

$$C_{0101}, C_{0102}, C_{0112}, C_{0202}, C_{0212}, C_{0303}$$

$$C_{0313}, C_{0323}, C_{1212}, C_{1313}, C_{1323}, C_{2323}.$$

$$\frac{r^4 \sin^2\theta}{e^{2\beta}} C_{1010} = e^{2\gamma}(V - r^4 e^{2\gamma-2a}) C_{1313} - 2r^2 e^{2\gamma} C_{0313}, \tag{25}$$

$$\frac{r^2 \sin^2\theta}{e^{2\gamma}} C_{0112} = e^{2\beta} C_{1323} - r^2 e^{2\gamma} C_{1313}, \tag{26}$$

$$\frac{2r^2 \sin^2\theta}{e^{2\gamma}} C_{0212} = e^{2\beta} C_{2323} - r e^{2\gamma} C_{1313}, \tag{27}$$

$$\sin^2\theta C_{1212} = -e^{4\gamma} C_{1313}. \tag{28}$$

$$\begin{aligned} & H^0_0 = H^0_1 = H^0_2 = H^1_0 = H^1_1 = H^1_2 = \\ & \quad = H^2_0 = H^2_1 = H^2_2 = H^3_0 = H^3_3 = 0 \end{aligned} \tag{29}$$

$$\begin{aligned} & H^0_3 = -\frac{1}{r}\left\{2c_u \cos\theta + c_{\theta u}\sin\theta\right\} \\ + & \frac{1}{r^2}\left\{4c_u(c-M)\cos\theta\right. \\ + & \left.\left[\frac{3}{2}(N_u + M_\theta + c_u c_\theta) + \frac{7}{2}cc_{\theta u} - 2Mc_{\theta u}\right]\sin\theta\right\} \\ + & \frac{1}{r^3}\left\{-\frac{N}{\sin\theta}[1+2c_u]\right. \\ + & [8Mc_u(c-M) + N_\theta(1-2c_u) \\ + & \left.\frac{5}{2}c^2 c_u - Nc_{\theta u} - P_u - 4Mc\right]\cos\theta \end{aligned}$$

$$+ \quad \Big[2(N - Mc_\theta) - c_{\theta u}(7Mc - 4M^2 - N_\theta - \frac{7}{4}c^2)$$

$$+ \quad 3M(N_u + M_\theta) - \frac{1}{2}P_{\theta u}$$

$$- \quad 3cM_\theta + N_{\theta\theta} + c_u(8N + 3Mc_\theta + \frac{5}{2}cc_\theta)\Big] \sin\theta\Big\} \tag{30}$$

$$H_3^1 = \frac{1}{r}\{(c_\theta c_{uu} + c_{\theta u})\sin\theta + 2(cc_{uu} + c_u)\cos\theta\}$$

$$+\frac{1}{r^2}\Bigg\{\frac{4cc_u\cos\theta}{\sin^2\theta} + \frac{2c_u c_\theta - cc_{\theta u}}{\sin\theta}$$

$$+\Big[-\frac{1}{2}c_\theta c_{\theta\theta u} + (2M - 3c)c_\theta c_{uu} - \frac{5}{2}(cc_{\theta u} + c_u c_\theta)$$

$$-2Nc_{uu} - \frac{3}{2}(N_u + M_\theta)\Big]\sin\theta$$

$$+\Big[-6cc_u + 4c(M - c)c_{uu} - \frac{1}{2}c_\theta c_{\theta u} - cc_{\theta\theta u}\Big]\cos\theta\Bigg\}$$

$$+\frac{1}{r^3}\Bigg\{\frac{8cc_u(M - c)\cos\theta}{\sin^2\theta} + \frac{1}{\sin\theta}[c_u c_\theta(4M - 3c) + N - 2cN_u$$

$$+2Ncc_{uu} + cc_{\theta u}(7c - 2M) - 4c_u N - cM_\theta] +$$

$$\Big[\frac{1}{2}P_{\theta u} - N_{\theta\theta} - 2N + 4cM_\theta - 4c_u N + 2cN_u$$

$$+c_u c_\theta(-\frac{9}{2}c - M + cc_u + \frac{1}{2}c_{\theta\theta}) + c_\theta(2M - cM_u + P_{uu} + N_{\theta u} + \frac{1}{2}M_{\theta\theta})$$

$$+c_{\theta u}(-\frac{19}{4}c^2+2c_\theta^2+2Mc)+c_{\theta\theta u}(N+3cc_\theta-Mc_\theta)$$

$$+c_\theta c_{uu}(-6Mc+\frac{1}{2}c^2+N_\theta+4M^2)-2Nc_{uu}(c+2M)\Big]\sin\theta$$

$$+\Big[2c(2M+N_{\theta u}+P_{uu}-cM_u+\frac{1}{2}M_{\theta\theta})$$

$$+c_u(-2Mc-\frac{3}{2}c^2+2c^2c_u+\frac{3}{2}c_\theta^2+cc_{\theta\theta})$$

$$+c_{\theta u}(N-Mc_\theta+8cc_\theta)+c_{uu}(c^3+2cN_\theta+Nc_\theta-8Mc^2+8M^2c)$$

$$+cc_{\theta\theta u}(5c-2M)-c_\theta(\frac{1}{2}M_\theta+N_u)+P_u-N_\theta\Big]\cos\theta\Big\} \quad (31)$$

$$H_3^2=\frac{1}{r}\left\{\sin\theta c_{uu}\right\}$$

$$+\frac{1}{r^2}\left\{\frac{2c_u}{\sin\theta}-\frac{1}{2}c_{\theta u}\cos\theta+\left[2c_{uu}(M-c)-c_u-\frac{1}{2}c_{\theta\theta u}\right]\sin\theta\right\}$$

$$+\frac{1}{r^3}\Big\{4c_u\frac{M-c}{\sin\theta}+\Big[\frac{3}{2}c_uc_\theta-N_u-\frac{1}{2}M_\theta$$

$$+Nc_{uu}+(\frac{7}{2}c-M)c_{\theta u}\Big]\cos\theta+$$

$$\Big[(\frac{5}{2}c-M)c_{\theta\theta u}+(\frac{1}{2}c_{\theta\theta}-M)c_u+2c_\theta c_{\theta u}$$

$$+c_{uu}(2c^2+4M^2-4Mc+N_\theta)+$$

$$\frac{1}{2}M_{\theta\theta} - cM_u + N_{\theta u} + P_{uu} + cc_u^2\Big]\sin\theta\Big\} \tag{32}$$

$$H_1^3 = \frac{1}{r^3}\frac{2c_u\cos\theta + c_{\theta u}\sin\theta}{\sin^2\theta} \tag{33}$$

$$H_2^3 = \frac{1}{r}\frac{c_{uu}}{\sin\theta}$$
$$+\frac{1}{r^2\sin\theta}\left\{2c_{uu}(c+M) - c_u - \frac{1}{2}c_{\theta\theta u} - \frac{1}{2}\cot\theta c_{\theta u} + 2\frac{c_u}{\sin^2\theta}\right\}$$
$$+\frac{1}{r^3\sin\theta}\left\{\frac{4Mc_u}{\sin^2\theta}\right.$$
$$+\cot\theta\left[-N_u + Nc_{uu} - \frac{1}{2}c_uc_\theta - (\frac{1}{2}c + M)c_{\theta u} - \frac{1}{2}M_\theta\right]$$
$$+c_{uu}(N_\theta + 4Mc + 4M^2 + 2c^2) + cc_u^2 + (\frac{1}{2}c_{\theta\theta} - M)c_u +$$
$$\left.(\frac{1}{2}c - M)c_{\theta\theta u} + P_{uu} + \frac{1}{2}M_{\theta\theta} + c_\theta c_{\theta u} - cM_u + N_{\theta u}\right\} \tag{34}$$

$$E_0^0 = E_3^0 = E_0^1 = E_3^1 = E_0^2 = E_3^2 = E_0^3 = E_1^3 = E_2^3 = 0 \tag{35}$$

$$E_1^0 = \frac{2(cc_u + M)}{r^3} \tag{36}$$

$$E_2^0 = \frac{2c_u\cos\theta + c_{\theta u}\sin\theta}{r\sin\theta}$$

$$+\frac{1}{2\sin\theta r^2}\left\{8Mc_u\cos\theta+[c_{\theta u}(4M-3c)-3(M_\theta+c_uc_\theta+N_u)]\sin\theta\right\}$$

$$+\frac{1}{4r^3}\Big\{[1+2c_u]\frac{4N}{\sin^2\theta}+\cot\theta\left[4\left(Nc_{\theta u}+P_u-N_\theta\right)\right.$$
$$\left.+\,c_u\left(32M^2+8N_\theta-42c^2\right)\right]$$
$$+4\left(N-3cN_u-N_{\theta\theta}\right)-12M\left(M_\theta+N_u\right)$$
$$+c_{\theta u}\left(4N_\theta+16M^2-13c^2-12Mc\right)$$
$$-c_u\left(30cc_\theta+12Mc_\theta+32N\right)+2P_{\theta u}\Big\} \qquad (37)$$

$$E^1_1=-\frac{2cc_u(1+\cos^2\theta)}{r^3\sin^2\theta}+\frac{(c_\theta c_{\theta u}+2M)}{r^3}+\frac{2\cos\theta(c_uc_\theta+cc_{\theta u})}{r^3\sin\theta} \qquad (38)$$

$$E^1_2=-\frac{2(cc_{uu}+c_u)\cos\theta+(c_\theta c_{uu}+c_{\theta u})\sin\theta}{r\sin\theta}$$

$$+\frac{1}{2r^2}\Big\{3(N_u+M_\theta)+c_{uu}(2cc_\theta-4Mc_\theta+4N)+5c_\theta c_u+c_\theta c_{\theta\theta u}+$$
$$cc_{\theta u}+\cot\theta\left[2c(c_{\theta\theta u}+2c_u-4Mc_{uu})+c_\theta c_{\theta u}\right]$$
$$+\frac{2}{\sin^2\theta}(cc_{\theta u}-2c_\theta c_u)-\frac{8}{\sin^3\theta}cc_u\cos\theta\Big\}$$

$$-\frac{1}{r^3}\Big\{8cc_u(M-c)\frac{\cos\theta}{\sin^3\theta}+\frac{1}{\sin^2\theta}\left[(4M-7c)c_uc_\theta-4c_uN\right.$$

$$+2Ncc_{uu} + (c^2 - 2Mc)c_{\theta u} + N - c(M_\theta + 2N_u)\Big]$$

$$+ \cot\theta \Big[c_u(-\frac{1}{2}c_\theta^2 - 2Mc + cc_{\theta\theta} + 2c^2 c_u - \frac{15}{2}c^2)$$

$$c_\theta \Big(c_{uu}N + (3c - M)c_{\theta u} - N_u - \frac{1}{2}M_\theta\Big)$$

$$+cc_{uu}(8M^2 - 3c^2 + 2N_\theta) + cc_{\theta\theta u}(3c - 2M)$$

$$+c_{\theta u}N + cM_{\theta\theta} + P_u - N_\theta + 2c(P_{uu} + N_{\theta u} - M - cM_u)\Big]$$

$$+c_{uu}\Big(c_\theta(-\frac{7}{2}c^2 + N_\theta - 2Mc + 4M^2) - 6Nc - 4MN\Big)$$

$$+c_u\Big(c_\theta(\frac{1}{2}c_{\theta\theta} - \frac{9}{2}c - M + cc_u) - 4N\Big)$$

$$+c_{\theta u}(c_\theta^2 + 2Mc - \frac{15}{4}c^2) + c_{\theta\theta u}(N - Mc_\theta + 2cc_\theta)$$

$$+c_\theta(\frac{1}{2}M_{\theta\theta} - M + N_{\theta u} + P_{uu} - M_u c)$$

$$-N_{\theta\theta} + \frac{1}{2}P_{\theta u} - cN_u + N + cM_\theta\Big\} \qquad (39)$$

$$E_1^2 = -\frac{2c_u\cos\theta + c_{\theta u}\sin\theta}{r^3\sin\theta} \qquad (40)$$

$$E_2^2 = -\frac{c_{uu}}{r} + \frac{1}{2r^2}\Big\{c_{\theta\theta u} - 4Mc_{uu} + 2c_u + \cot\theta c_{\theta u} - \frac{4c_u}{\sin^2\theta}\Big\}$$

$$
+\frac{1}{r^3}\left\{M_u c - \frac{1}{2}M_{\theta\theta} + M - N_{\theta u} - P_{uu}\right.
$$

$$
+\cot\theta\left[Mc_{\theta u} + \frac{1}{2}M_\theta + N_u - \frac{1}{2}cc_{\theta u} + \frac{1}{2}c_\theta c_u - Nc_{uu}\right]
$$

$$
+c_u\left(\frac{4(c-M)}{\sin^2\theta} + M - c - cc_u - \frac{1}{2}c_{\theta\theta}\right)
$$

$$
\left.-c_{uu}(4M^2 + N_\theta) - c_\theta c_{\theta u} + c_{\theta\theta u}(M - \frac{3}{2}c)\right\} \tag{41}
$$

$$
E_3^3 = \frac{c_{uu}}{r} - \frac{1}{2r^2}\left\{c_{\theta\theta u} - 4Mc_{uu} + 2c_u + \cot\theta c_{\theta u} - \frac{4c_u}{\sin^2\theta}\right\}
$$

$$
+\frac{1}{r^3}\left\{M + N_{\theta u} + P_{uu} + \frac{1}{2}M_{\theta\theta} - cM_u\right.
$$

$$
+\cot\theta\left[\frac{5}{2}cc_{\theta u} - \frac{1}{2}M_\theta - N_u + Nc_{uu} - Mc_{\theta u} + \frac{3}{2}c_u c_\theta\right]
$$

$$
+c_u\left(\frac{4M}{\sin^2\theta} + cc_u - M + \frac{1}{2}c_{\theta\theta} - c\right)
$$

$$
\left.+c_{uu}(4M^2 + N_\theta) + 2c_\theta c_{\theta u} + (\frac{3}{2}c - M)c_{\theta\theta u}\right\} \tag{42}
$$

$$
P \equiv C - \frac{c^3}{6}. \tag{43}
$$

$$
Q = H^\alpha_\beta E^\beta_\alpha, \qquad L = E^\alpha_\beta E^\beta_\alpha - H^\alpha_\beta H^\beta_\alpha. \tag{44}
$$

$$Q = 0 \tag{45}$$

$$L = \frac{2}{r^6}\left\{3(cc_u + M)^2 + (c^3 + 6P)c_{uu} + 6N(c_{\theta u} + 2c_u \cot\theta)\right\} + O(1/r^7) \tag{46}$$

3 Vorticity and gravitational radiation

4 Discussion

- What consequences do emerge from the vanishing of the magnetic part of the Weyl tensor?.
- What consequences do emerge from the vanishing of the electric part of the Weyl tensor?.
- How do different types of fields (radiative, non radiative but time dependent, and static) enter into the electric and magnetic part of the Weyl tensor, and into the corresponding invariants?.
- Why does gravitational radiation produce vorticity?.

$H^{\alpha}_{\beta} = 0$

$$c_{\theta u} \sin\theta + 2c_u \cos\theta = 0 \tag{47}$$

$$c_u = 0. \tag{48}$$

$$M_\theta = N_u = 0, \tag{49}$$

$$N_{\theta\theta} \sin^2\theta + N_\theta \sin\theta\cos\theta - N\cos 2\theta - (2cM\sin^2\theta)_\theta = 0 \tag{50}$$

$$N = \left(\int \frac{2cM}{\sin\theta}\, d\theta + \sigma\right)\sin\theta \tag{51}$$

$$C_u = 0. \tag{52}$$

$E^\alpha_\beta = 0$

$$c_u = 0, \tag{53}$$

$M = 0.$